Risk Measures, Stochastic Orders and Comonotonicity

Jan Dhaene

Sums of r.v.'s

- Many problems in risk theory involve sums of r.v.'s:

$$S = X_1 + X_2 + \cdots + X_n.$$

- Standard techniques for (approximate) evaluation of the d.f. of S are: convolution, moment-based approximations, De Pril's recursion, Panjer's recursion.
- Assuming independence of the X_i is convenient but not always appropriate.
- The copula approach: (Frees & Valdez, 1998)

$$\Pr\left[X_1 \leq x_1, \cdots, X_n \leq x_n\right] = C\left[F_{X_1}(x_1), \ldots, F_{X_n}(x_n)\right].$$

Sums of r.v.'s

- **Problem to solve**:
  Determine risk measures related to $S = X_1 + X_2 + ... + X_n$ in case $F_{X_i}(x)$ known, C complicated / unknown, positive dependence.
- **How to solve**?
  - Derive stochastic lower and upper bounds for S:

$$S^l \precsim S \precsim S^u.$$

  - Approximate $R[S]$ by $R[S^u]$ or $R\left[S^l\right]$.
  - Involves **risk measurement**, **stochastic orderings**, **choice under risk**.

Example: A life annuity

- Consider a life annuity a_x on (x) with present value

$$S = X_1 + X_2 + \cdots + X_n,$$

where

$$X_j = \begin{cases} 0 & : T(x) \leq j, \\ v^j & : T(x) > j. \end{cases}$$

- All X_j are increasing functions of $T(x)$
$\Longrightarrow (X_1, X_2, \cdots, X_n)$ is comonotonic.

Example: A risk sharing scheme

- Consider a loss $S \geq 0$ that is covered by n parties:

$$S = X_1 + X_2 + \cdots + X_n.$$

- X_j is the layer with deductible a_j and maximal payment $(a_{j+1} - a_j)$:

$$X_j = \begin{cases} 0 & : 0 \leq S \leq a_j \\ S - a_j & : a_j < S \leq a_{j+1} \\ a_{j+1} - a_j & : S > a_{j+1}, \end{cases}$$

  with $a_0 = 0$ and $a_{n+1} = \infty$.

- The layers X_j are comonotonic.
- Many risk sharing schemes lead to partial risks that are comonotonic.

Example: A portfolio of pure endowments

- Consider a portfolio $(X_1, X_2, \cdots, X_n)$ of m-year pure endowment insurances.
- X_j: claim amount of policy j at time m:

$$X_j = \begin{cases} 0 & : T(x) \leq m, \\ 1 & : T(x) > m. \end{cases}$$

- Assumption: all X_i are i.i.d. and $X_i \stackrel{d}{=} X$.

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Example: A portfolio of pure endowments

- Let
$$S = (X_1 + X_2 + \cdots + X_n)\ Y,$$
where Y is the stochastic discount factor over $[0, m]$.
- Assumption: X_i and Y are mutually independent.
- Risk pooling reduces the actuarial risk, not the financial risk:
$$\mathrm{Var}\left[\frac{S}{n}\right] = \frac{\mathrm{Var}[X]}{n}\ \mathrm{E}[Y^2] + (\mathrm{E}[X])^2\ \mathrm{Var}[Y]$$
$$\to (\mathrm{E}[X])^2\ \mathrm{Var}[Y].$$
- The terms $X_i Y$ are 'conditionally comonotonic'.

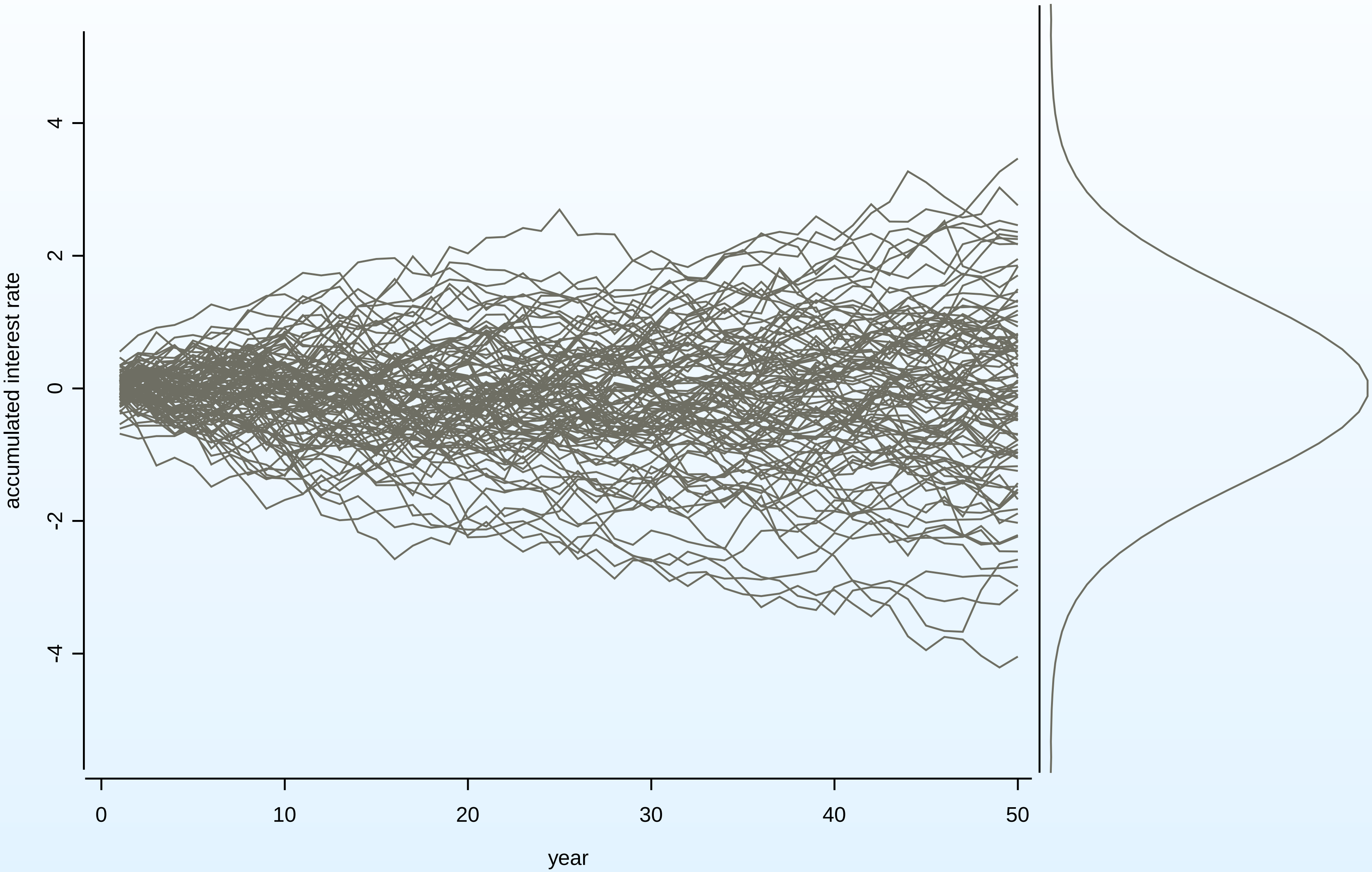
Example: A provision for future liabilities

- Consider the liability cash-flow stream $(\alpha_1, 1), (\alpha_2, 2), \ldots, (\alpha_n, n)$.
- The provision is invested such that it generates a cumulative log-return $Y(i)$ over the period $(0, i)$.
- The provision is determined as $R[S]$ with

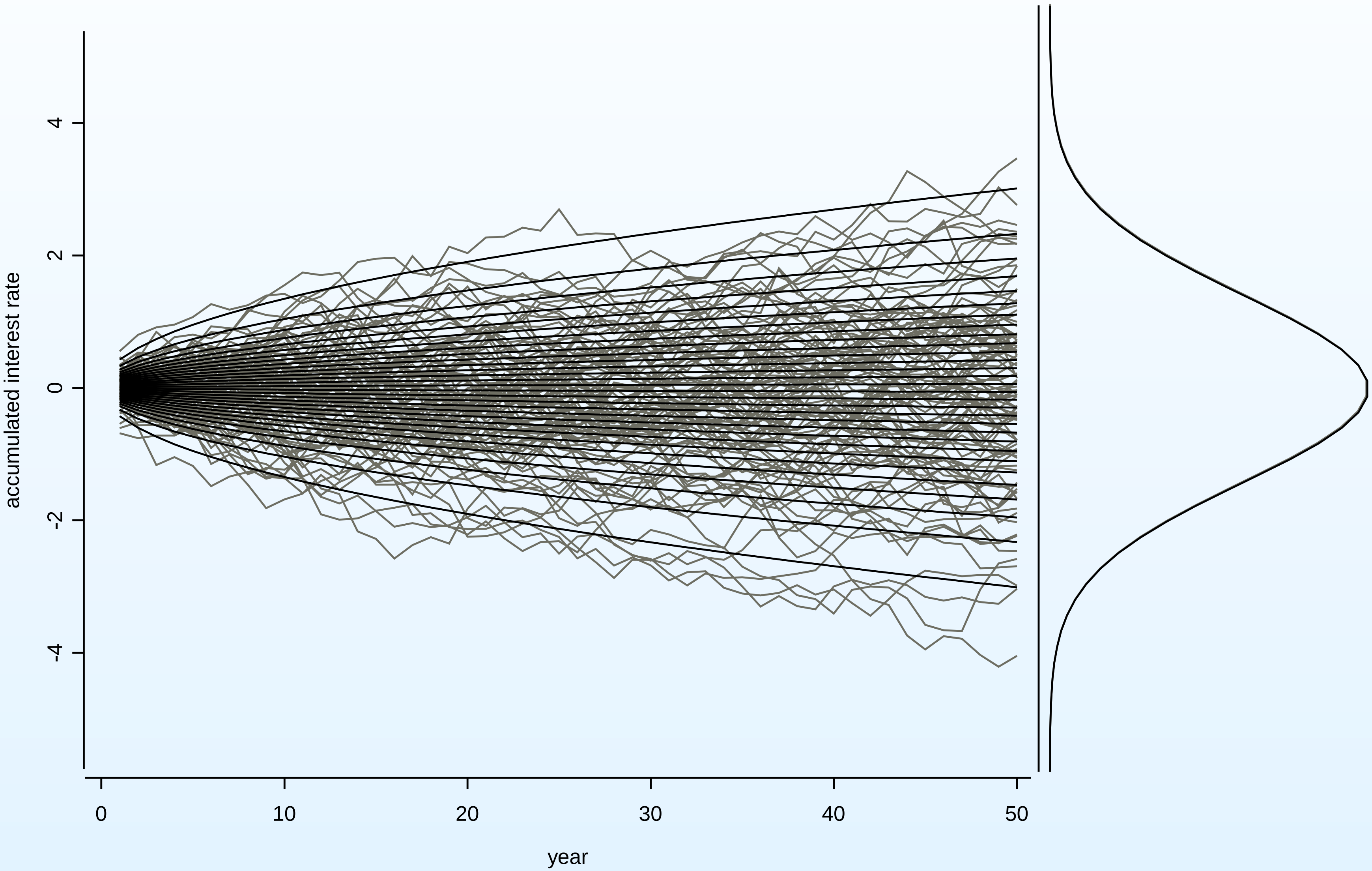
$$S = \sum_{i=1}^{n} \alpha_i \, e^{-Y(i)}.$$

- The cumulative returns $Y(i)$ are 'locally quasi-comonotonic'.
- Illustration: i.i.d. normal yearly returns.

Cumulative returns



Cumulative returns



Comonotonicity

- Definition: $(X_1, \cdots, X_n)$ is **comonotonic** if there exists a r.v. Z and increasing functions $f_1, \cdots, f_n$ such that

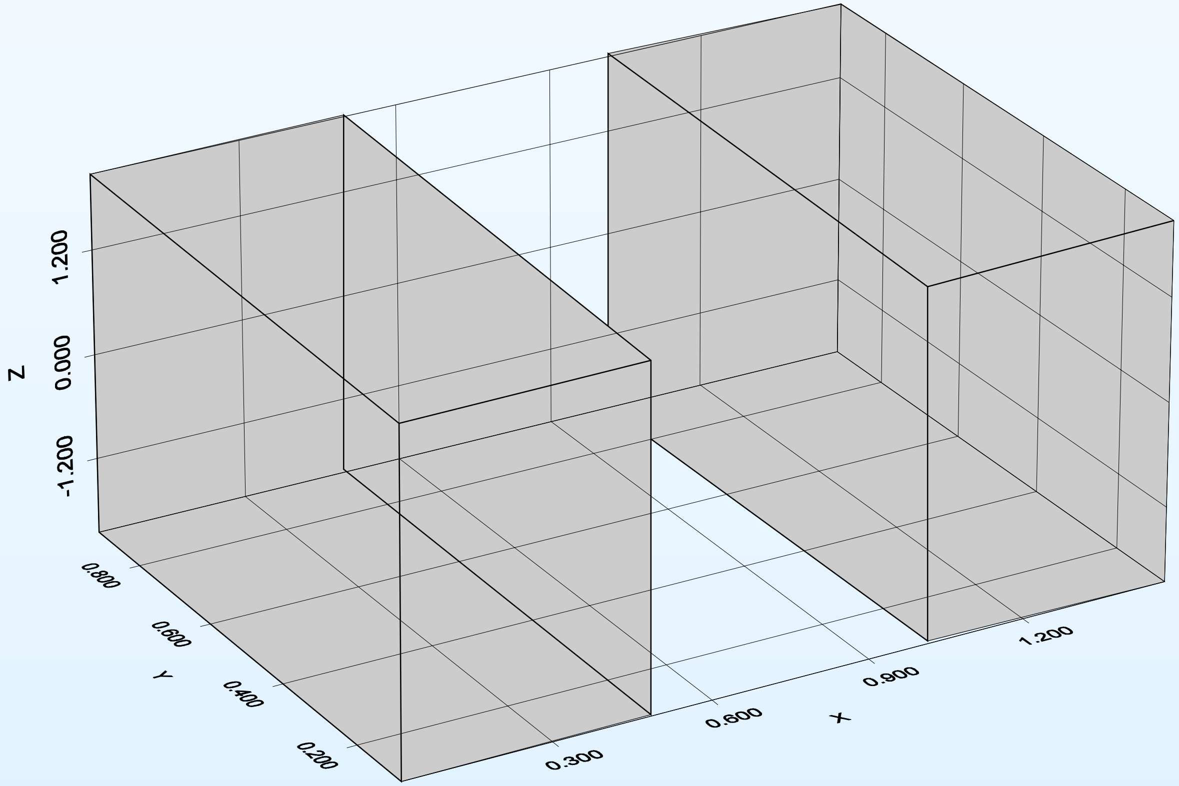
$$(X_1, \cdots, X_n) \stackrel{d}{=} (f_1(Z), \cdots, f_n(Z)) .$$

- Determining the d.f. of $(X_1, \cdots, X_n)$ is a one-dimensional problem.
- Comonotonicity is very strong positive dependency structure.
- Adding comonotonic r.v.'s produces no diversification: If all X_i are identically distributed and comonotonic, then

$$\frac{X_1 + \cdots + X_n}{n} \stackrel{d}{=} X_1 .$$

An example of comonotonic r.v.'s

- Consider (X, Y, Z) with
  $X \sim$ Uniform on $(0, \frac{1}{2}) \cup \left(1, \frac{3}{2}\right)$
  $Y \sim$ Beta (2,2)
  $Z \sim$ Normal (0,1).
- (X, Y, Z) mutually independent



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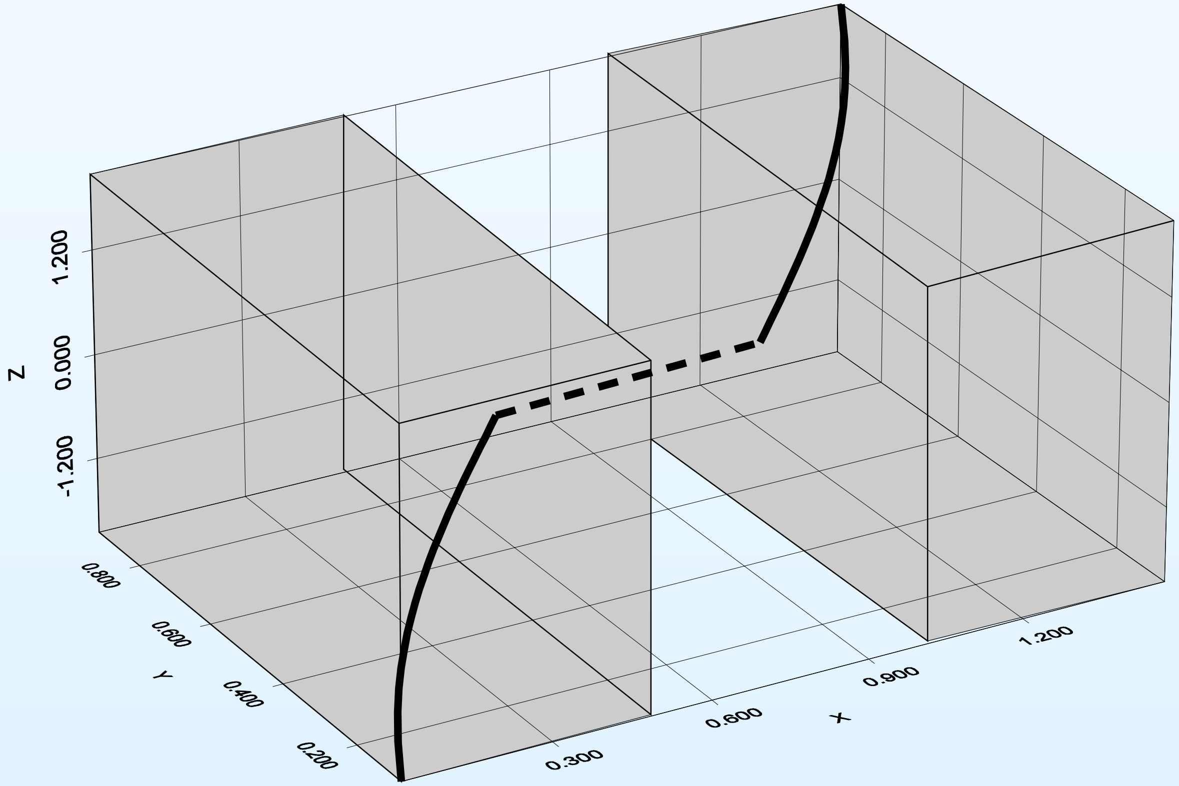
- (X, Y, Z) comonotonic



Sums of comonotonic r.v.'s

- Notation: $(X_1^c, \ldots, X_n^c)$ is comonotonic and has same marginals as $(X_1, \ldots, X_n)$.
- $S^c = X_1^c + X_2^c + \cdots + X_n^c$.
- Quantiles of S^c:

$$F_{S^c}^{-1}(p) = \sum_{i=1}^{n} F_{X_i}^{-1}(p).$$

- Distribution function of S^c:

$$\sum_{i=1}^{n} F_{X_i}^{-1}\left[F_{S^c}(x)\right] = x.$$

Sums of comonotonic r.v.'s

- Stop-loss premiums of S^c:
  (Dhaene, Wang, Young, Goovaerts, 2000)

$$\mathrm{E}\left[S^c - d\right]_+ = \sum_{i=1}^{n} \mathrm{E}\left[\left(X_i - d_i\right)_+\right]$$

  with

$$d_i = F_{X_i}^{-1}\left[F_{S^c}(d)\right].$$

- Application: (Jamshidian, 1989)
  In the Vasicek (1977) model, the price of a European call option on a coupon bond = sum of the prices of European options on zero coupon bonds.

Theories of choice under risk

- Wealth level vs. probability level:

$$\mathrm{E}[X] = \int_0^1 F_X^{-1}(1-q)\, dq.$$

- Utility theory: (von Neumann & Morgenstern, 1947)

$$\mathrm{E}[u(X)] = \int_0^1 u\left[F_X^{-1}(1-q))\right]\, dq$$

with $u(x)$ a utility function.

- Dual theory of choice under risk: (Yaari, 1987)

$$\rho_f[X] = \int_0^1 F_X^{-1}(1-q)\, df(q)$$

with $f(q)$ a distortion function.

Theories of choice under risk

- Choice under risk:
  - Prefer wealth Y over wealth X if
    $$\mathrm{E}\left[u(X)\right] \leq \mathrm{E}\left[u(Y)\right].$$
  - Prefer wealth Y over wealth X if
    $$\rho_f\left[X\right] \leq \rho_f\left[Y\right].$$
- Additivity:
  - If $u(0) = 0$ and $\Pr\left[X \neq 0,\ Y \neq 0\right] = 0$, then
    $$\mathrm{E}\left[u(X+Y)\right] = \mathrm{E}\left[u(X)\right] + \mathrm{E}\left[u(Y)\right].$$
  - If X and Y are comonotonic, then
    $$\rho_f\left[X+Y\right] = \rho_f\left[X\right] + \rho_f\left[Y\right].$$

Risk measures

- Definition:
  = mapping from the set of quantifiable losses to the real line:

$$X \to R[X].$$

- Have been investigated extensively in the literature:
  - Huber (1981):
    upper expectations,
  - Goovaerts, De Vylder & Haezendonck (1984):
    premium principles,
  - Artzner, Delbaen, Eber & Heath (1999):
    coherent risk measures.

Construction of risk measures

- The equivalent expected utility principle:

$$u(w) = \mathrm{E}\left[u(w + R[X] - X)\right].$$

- The equivalent distorted expectation principle:

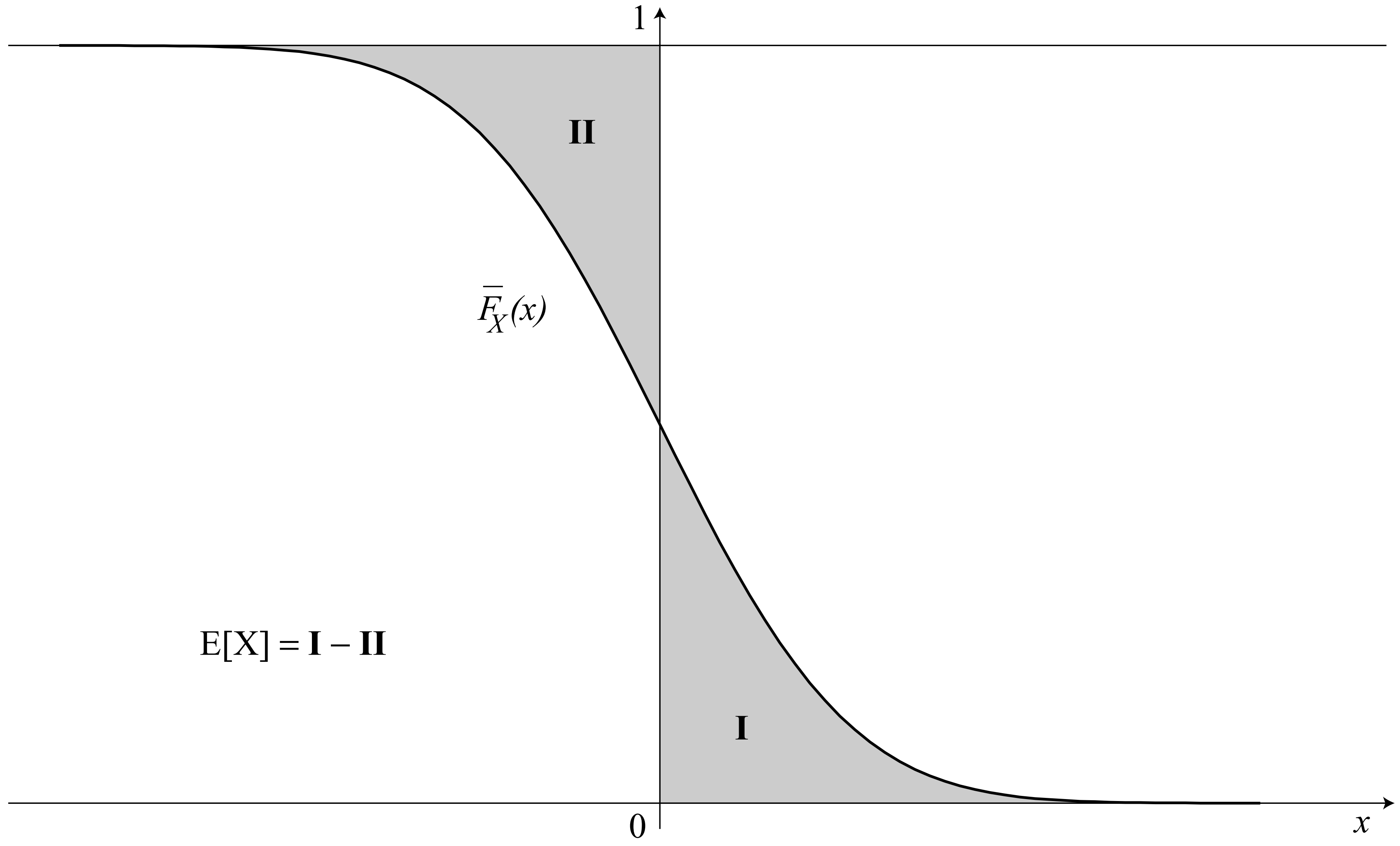
$$\rho_f[w] = \rho_f[w + R[X] - X].$$

This leads to distortion risk measures (Wang, 1996):

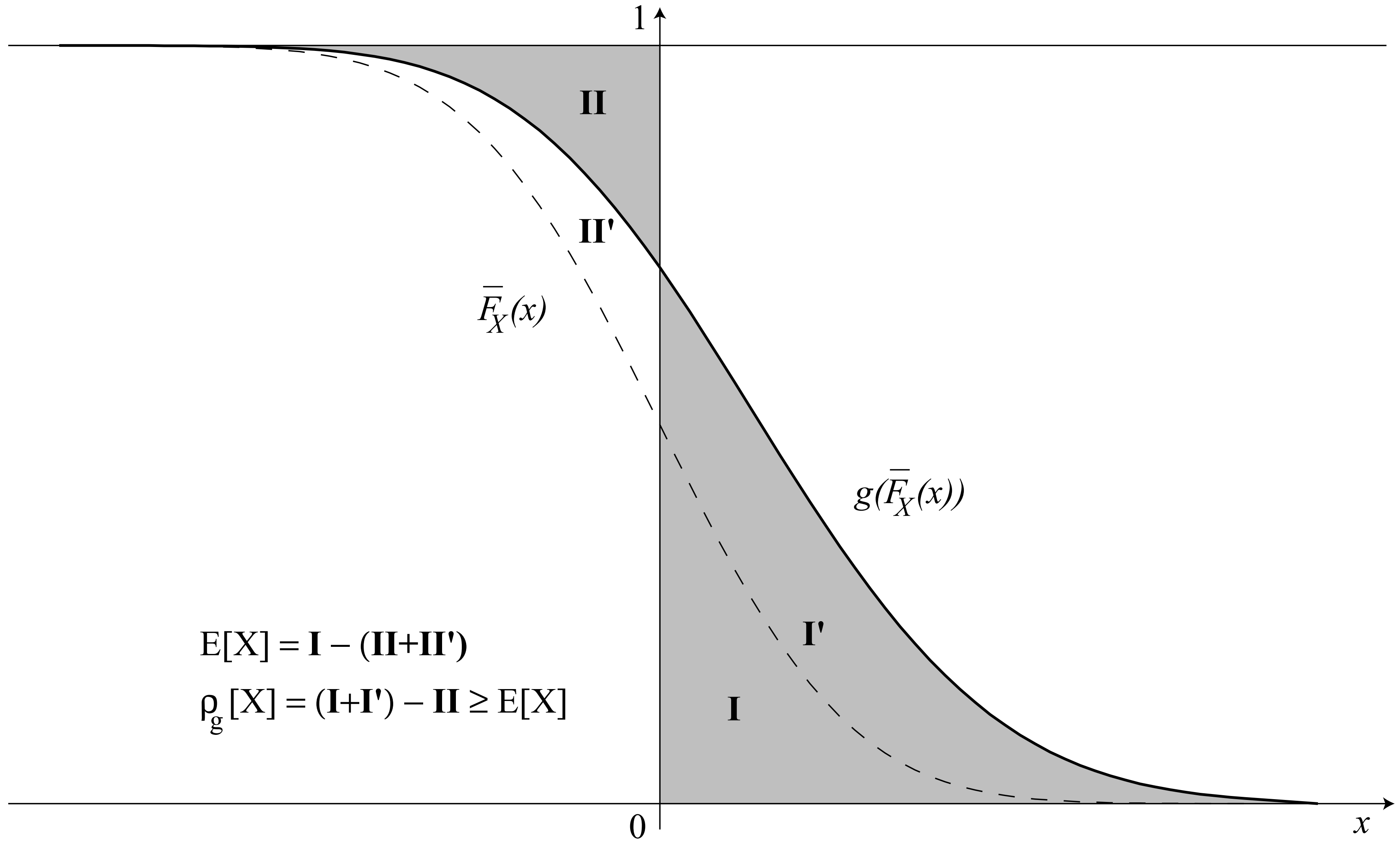
$$R[X] = \rho_g[X]$$

with $g(q) = 1 - f(1-q)$.

Distortion risk measures



Distortion risk measures



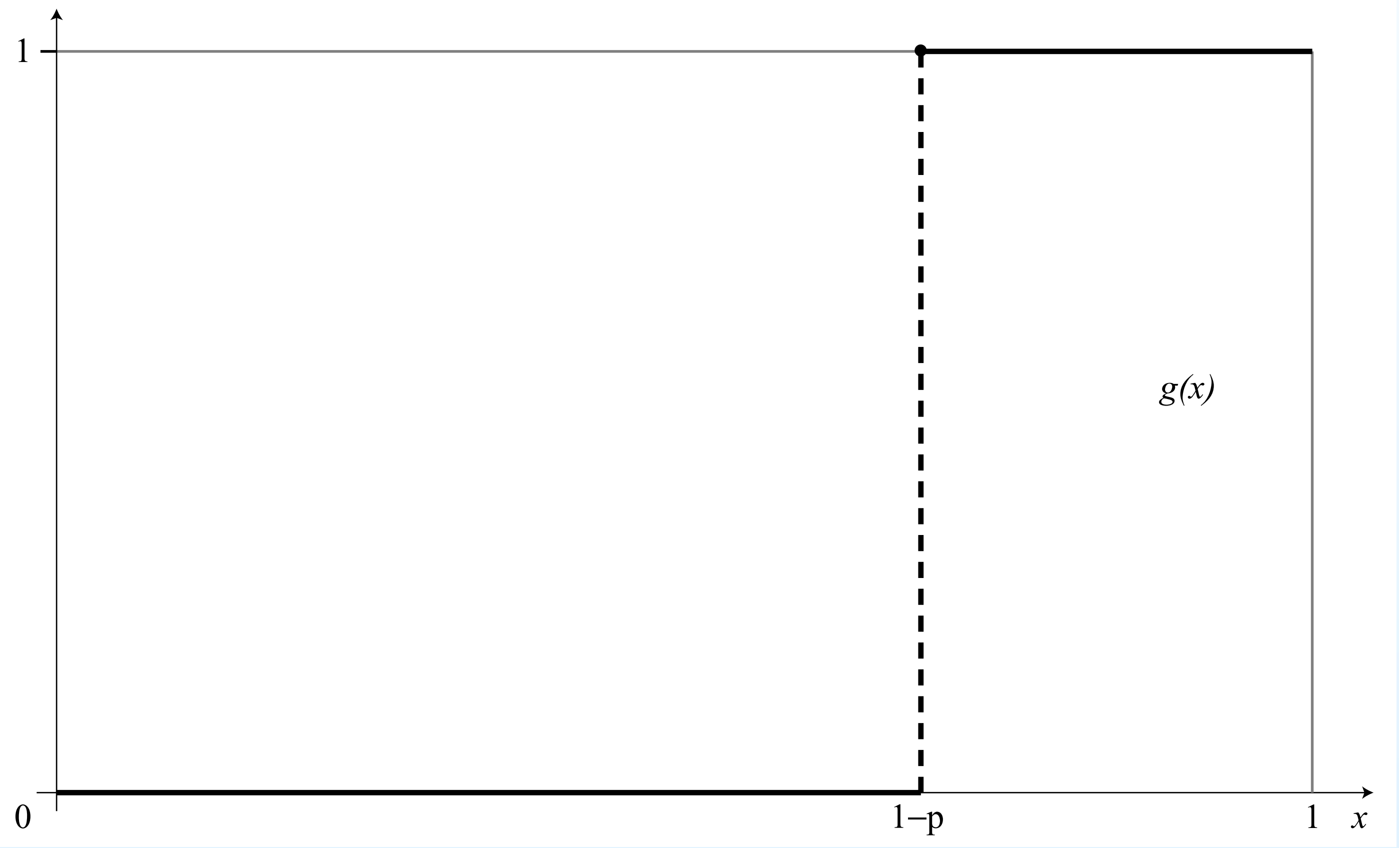
$E[X] = \mathbf{I} - (\mathbf{II}+\mathbf{II'})$

$\rho_g[X] = (\mathbf{I}+\mathbf{I'}) - \mathbf{II} \geq E[X]$

Examples of distortion risk measures

- Value-at-Risk: $X \to \mathrm{VaR}_p[X] = F_X^{-1}(p) = Q_p[X]$.

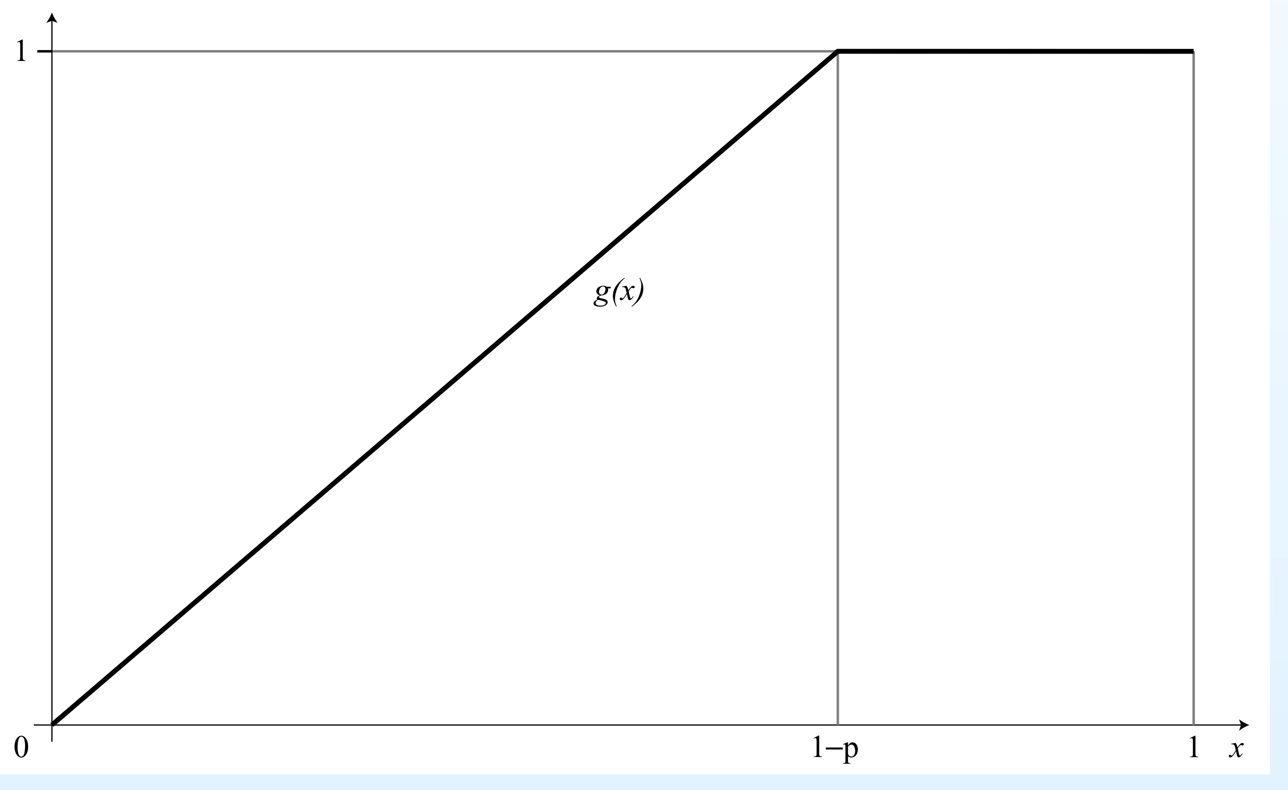
$$g(x) = I\,(x > 1-p)\,, \qquad 0 \leq x \leq 1.$$



Examples of distortion risk measures

- Tail Value-at-Risk: $X \rightarrow \text{TVaR}_p(X) = \frac{1}{1-p} \int_p^1 \text{VaR}_q\,[X]\;dq$.

$$g(x) = \min\left(\frac{x}{1-p}, 1\right), \qquad 0 \leq x \leq 1.$$



Concave distortion risk measures

- $\rho_g[.]$ is a concave distortion risk measure if g is concave.
- TVaR_p is concave, VaR_p not.
- Concave distortion risk measures are subadditive:

$$\rho_g[X+Y] \leq \rho_g[X] + \rho_g[Y].$$

- Optimality of TVaR_p:

$$\mathrm{TVaR}_p[X] = \min\left\{\rho_g([X] \mid g \text{ is concave and } \rho_g \geq \mathrm{VaR}_p\right\}.$$

- Optimality of VaR_p: (Artzner et al. 1999)

$$\mathrm{VaR}_p[X] = \inf\left\{\rho([X] \mid \rho \text{ is coherent and } \rho \geq \mathrm{VaR}_p\right\}.$$

Optimality of VaR_p

- Consider a loss X and a solvency capital requirement $R[X]$.
- Measuring the insolvency risk:

$$(X - R[X])_+ \longrightarrow \mathrm{E}\left[(X - R[X])_+\right].$$

- How to choose $R[X]$?
  - $\mathrm{E}\left[(X - R[X])_+\right]$ should be small
    $\Rightarrow$ choose $R[X]$ large enough.
  - Capital has a cost
    $\Rightarrow$ $R[X]$ should be small enough.

Optimality of VaR_p

- The optimal capital requirement:
  $R[X]$ is determined as the minimizer (with respect to d) of

$$\mathrm{E}\left[(X-d)_+\right] + d\,\varepsilon, \qquad 0 < \varepsilon < 1.$$

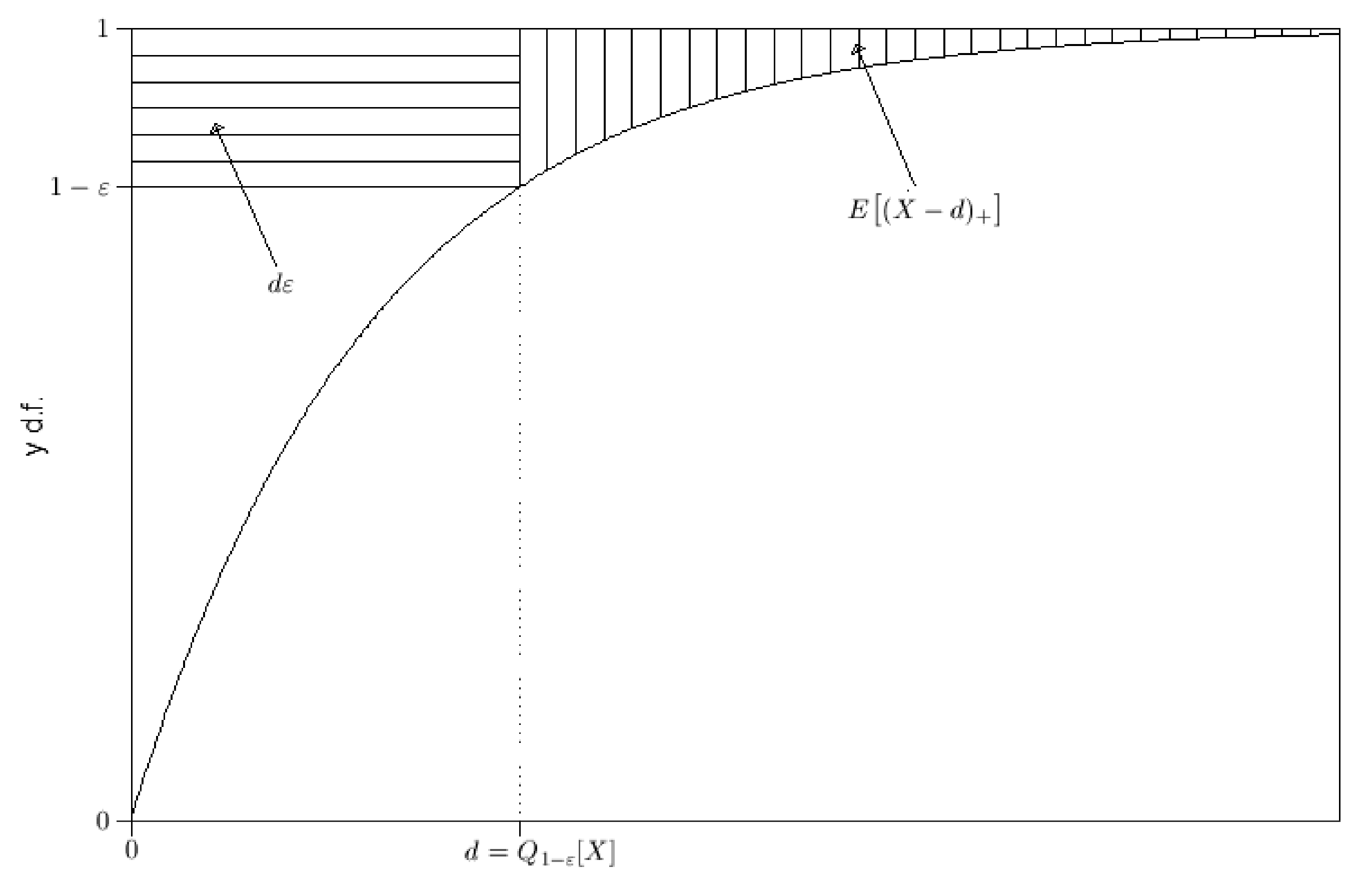
- Solution:

$$R[X] = \mathrm{VaR}_{1-\varepsilon}[X].$$

- The minimum is given by $\varepsilon\,\mathrm{TVaR}_{1-\varepsilon}[X]$.
- Geometric proof (for $\mathrm{VaR}_{1-\varepsilon}[X] > 0$):

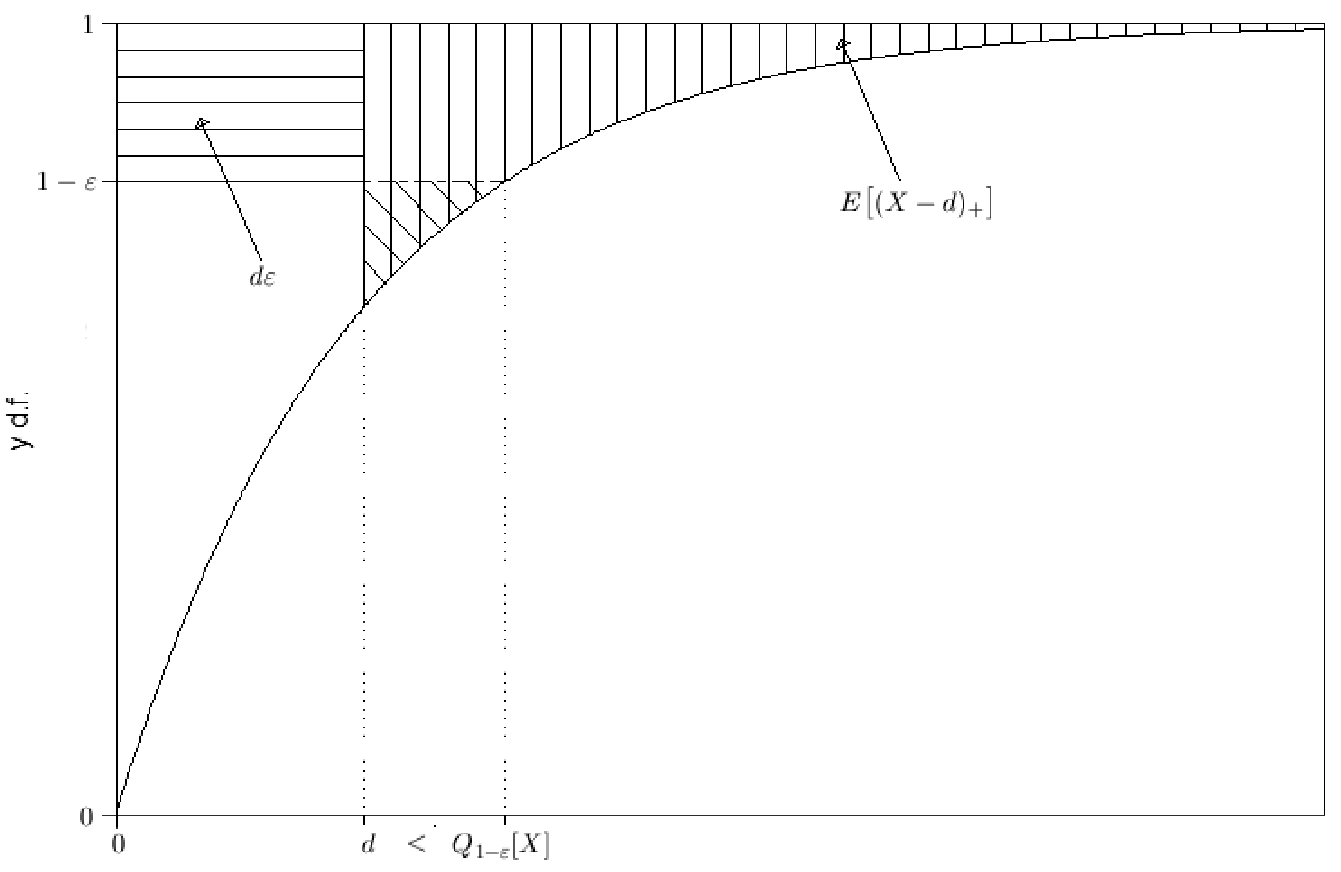
Optimality of VaR$_p$

$$\mathrm{E}\left[(X-d)_+\right] + d\,\varepsilon \text{ with } d = Q_{1-\varepsilon}[X]$$



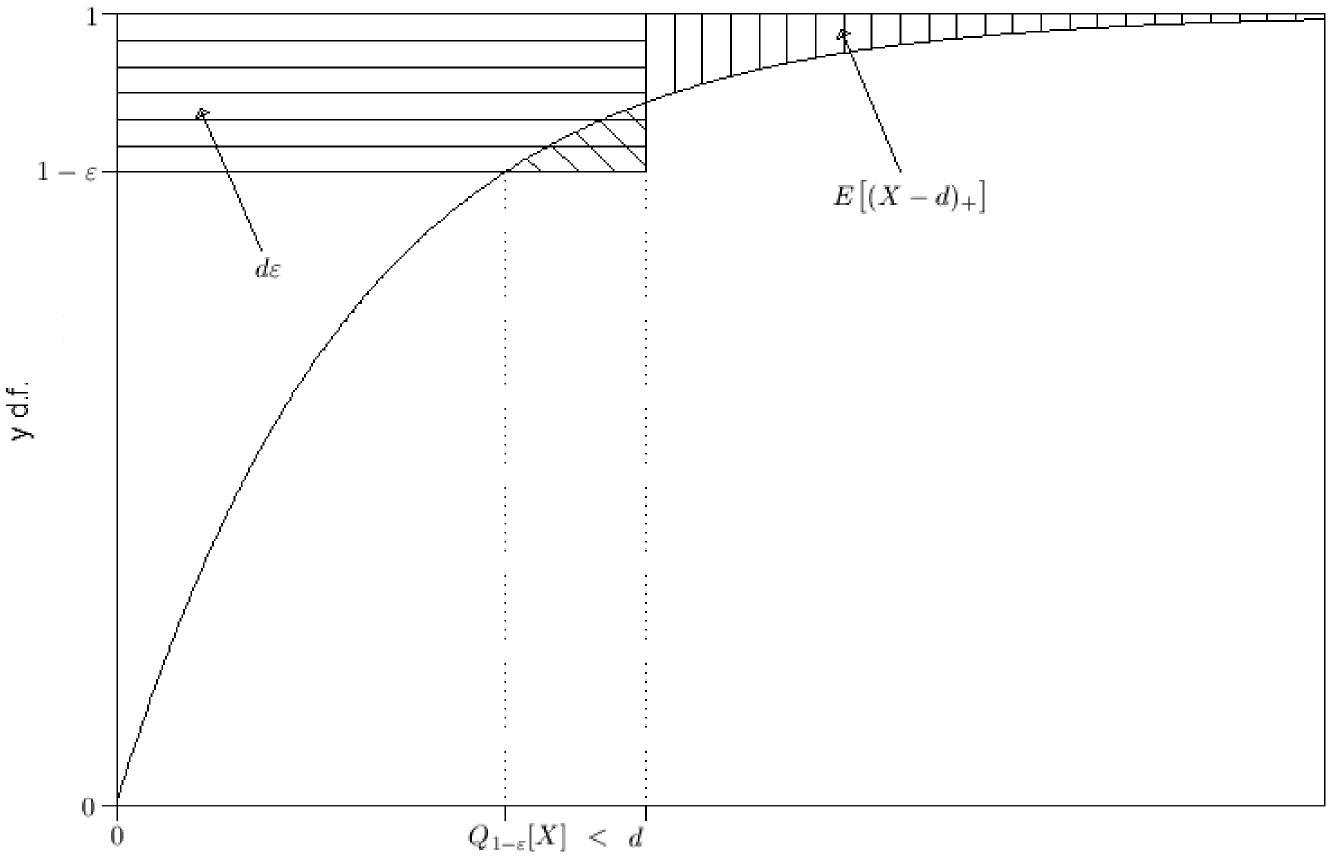
Optimality of VaR$_p$

$$\mathrm{E}\left[(X-d)_+\right] + d\,\varepsilon \text{ with } d < Q_{1-\varepsilon}[X]$$



Optimality of VaR$_p$

$$\mathrm{E}\left[(X-d)_+\right] + d\,\varepsilon \text{ with } d > Q_{1-\varepsilon}[X]$$



Can a risk measure be too subadditive?

(Dhaene, Laeven, Vanduffel, Darkiewicz, Goovaerts, 2005)

- For losses X and Y, we have that

$$
\begin{aligned}
&\mathrm{E}\left[(X+Y-\mathsf{R}[X]-\mathsf{R}[Y])_+\right] \\
&\qquad \leq \mathrm{E}\left[(X-\mathsf{R}[X])_+\right]+\mathrm{E}\left[(Y-\mathsf{R}[Y])_+\right].
\end{aligned}
$$

- Splitting increases the insolvency risk
  $\Rightarrow$ the risk measure used to determine the required solvency capital should be subadditive enough.
- Merging decreases the insolvency risk
  $\Rightarrow$ subadditivity of the capital requirement is allowed *to some extent*.
  $\Rightarrow$ the capital requirement can be *too subadditive* if no constraint is imposed on the subadditivity.

Can a risk measure be too subadditive?

- The *regulator's condition*:

$$\mathrm{E}\left[(X+Y-\mathsf{R}[X+Y])_+\right] + \varepsilon\,\mathsf{R}[X+Y]$$
$$\leq \mathrm{E}\left[(X-\mathsf{R}[X])_+\right] + \mathrm{E}\left[(X-\mathsf{R}[X])_+\right] + \varepsilon\left(\mathsf{R}[X]+\mathsf{R}[Y]\right)$$

- $\mathrm{VaR}_{1-\varepsilon}[\cdot]$ fulfills the regulator's condition.
- Any subadditive $R[\cdot] \geq \mathrm{VaR}_{1-\varepsilon}[\cdot]$ fulfills the regulator's condition.
- Mark(ovitz), 1959:
  'We might decide that in one context one basic set of principles is appropriate, while in another context a different set of principles should be used.'

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Stochastic orderings - Upper and lower tails

- $\mathrm{E}\left[(X-d)_+\right]$= surface above the d.f., from d on.
- $\mathrm{E}\left[(d-X)_+\right]$= surface below the d.f., from $-\infty$ to d.

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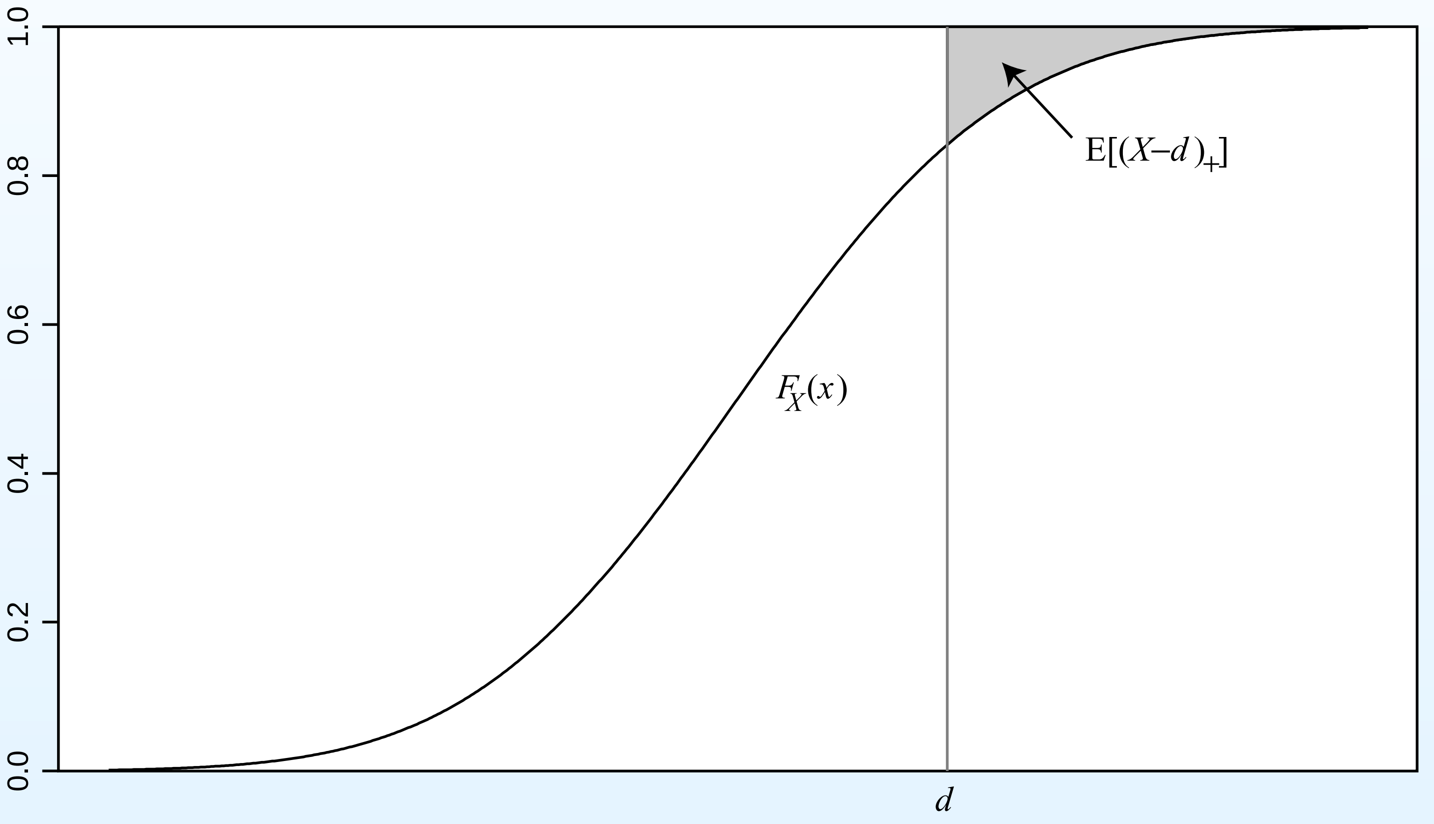

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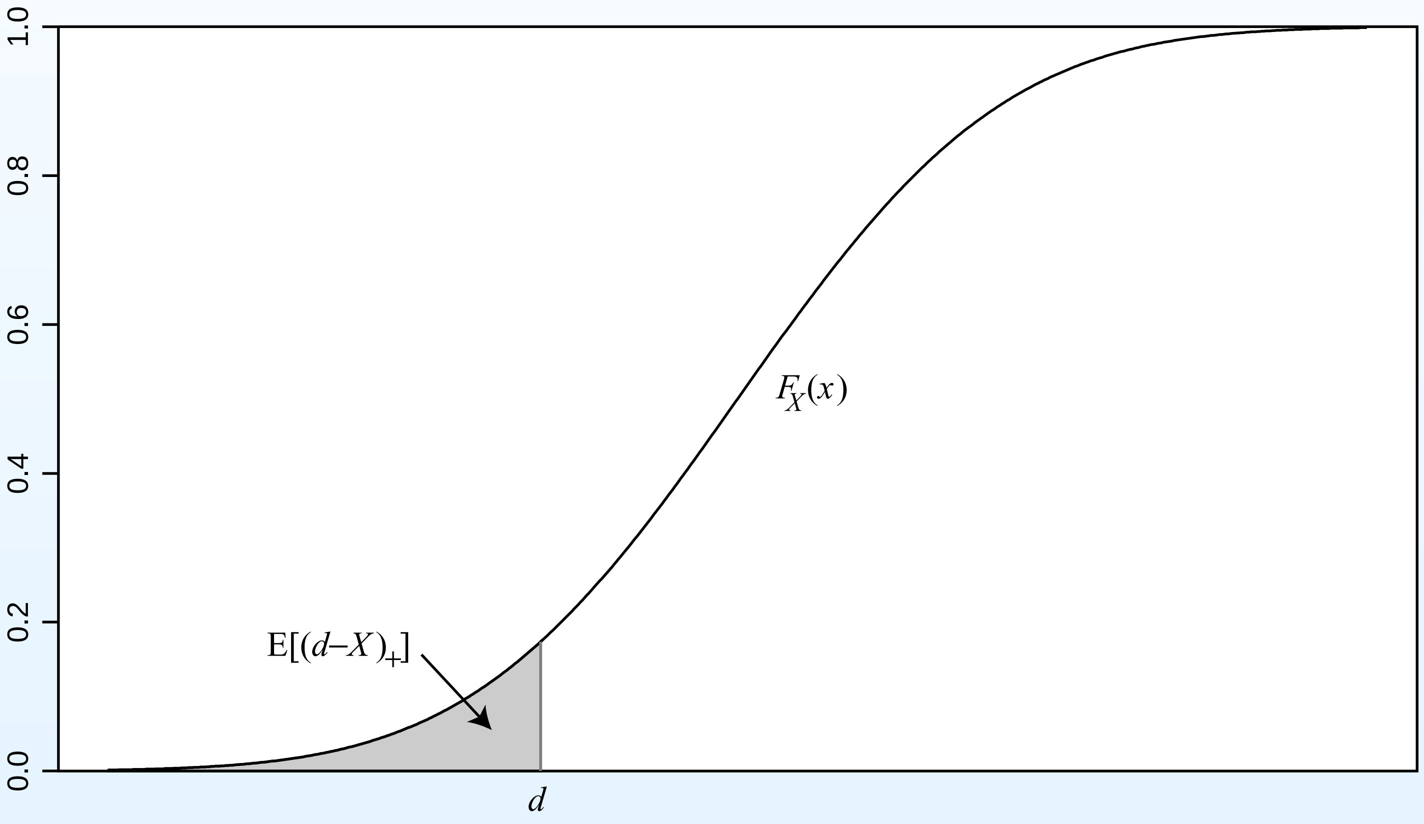

Convex order

- Definition:

$$X \leq_{cx} Y \Leftrightarrow \text{any tail of } Y \text{ exceeds the respective tail of } X.$$

- Represents common preferences of risk averse decision makers between r.v.'s with equal means.
- Characterization in terms of distortion risk measures: (Wang & Young, 1998)

$$X \leq_{cx} Y \Leftrightarrow \mathrm{E}[X] = \mathrm{E}[Y] \text{ and } \rho_g[X] \leq \rho_g[Y] \text{ for all concave } g.$$

Stochastic order bounds for sums of dependent r.v.'s

- Theorem: (Kaas et al., 2000)
  For any $(X_1, \cdots, X_n)$ and any Λ, we have that

$$\sum_{i=1}^{n} \mathrm{E}\left[X_i \mid \Lambda\right] \leq_{cx} \sum_{i=1}^{n} X_i \leq_{cx} \sum_{i=1}^{n} X_i^c$$

- Notation: $S^l \leq_{cx} S \leq_{cx} S^c$.
- Assume that all $\mathrm{E}\left[X_i \mid \Lambda\right]$ are $\nearrow$ functions of Λ
  $\Rightarrow S^l$ is a comonotonic sum.
- Why use these comonotonic bounds?
  - One-dimensional stochasticity.
  - $\rho_g\left[S^l\right]$ and $\rho_g\left[S^c\right]$ are easy to calculate.
  - If g is concave, then $\rho_g\left[S^l\right] \leq \rho_g\left[S\right] \leq \rho_g\left[S^c\right]$.

On the choice of Λ

(Vanduffel et al., 2004)

- Let

$$S = \sum_{i=1}^{n} \alpha_i \, e^{-Y(i)} \quad \text{and} \quad S^l = \sum_{i=1}^{n} \alpha_i \, \mathrm{E}\left[e^{-Y(i)} \mid \Lambda \right]$$

with $\alpha_i > 0$ and $(Y_1, \cdots, Y_n)$ normal.

- First order approximation for Var$\left[S^l\right]$:

$$\mathrm{Var}[S^l] \approx \mathrm{Corr}^2 \left[\sum_{i=1}^{n} \alpha_i \, \mathrm{E}[e^{-Y(i)}] Y(i), \Lambda \right] \mathrm{Var} \left[\sum_{i=1}^{n} \alpha_i \, \mathrm{E}[e^{-Y(i)}] Y(i) \right].$$

- Optimal choice for Λ:

$$\Lambda = \sum_{i=1}^{n} \alpha_i \; \mathrm{E}\left[e^{-Y(i)} \right] \; Y(i).$$

The continuous perpetuity

- Local comonotonicity: Let $B(\tau)$ be a standard Wiener process.
  The accumulated returns

  $$\exp\left[\mu\tau + \sigma\ B(\tau)\right] \text{ and } \exp\left[\mu\left(\tau + \Delta\tau\right) + \sigma\ B\left(\tau + \Delta\tau\right)\right]$$

  are 'almost comonotonic'.
- The continuous perpetuity: (Dufresne, 1989; Milevsky, 1997)

  $$S = \int_0^\infty \exp\left[-\mu\tau - \sigma\ B(\tau)\right]\ d\tau$$

  has a reciprocal Gamma distribution.

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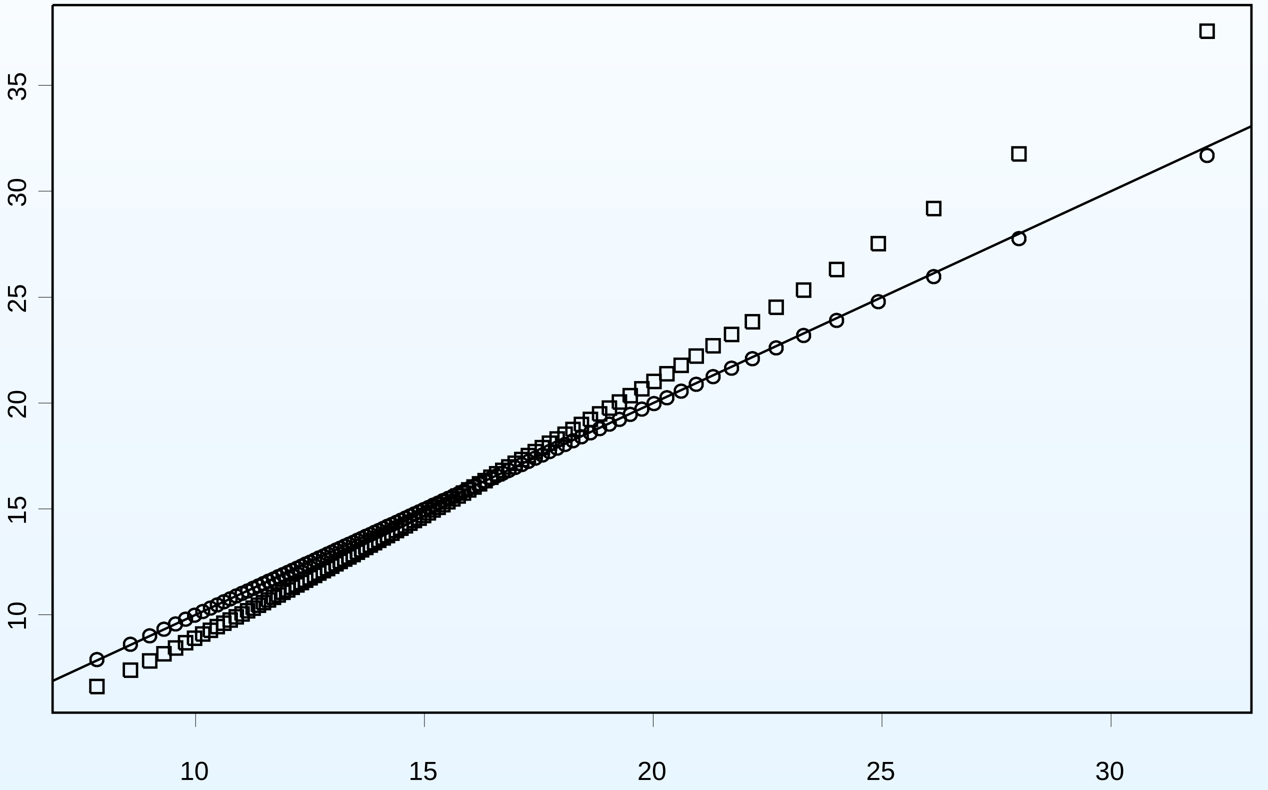
The continuous perpetuity

- Numerical illustration: $\mu = 0.07$ and $\sigma = 0.1$.



Squares = $(Q_p[S], Q_p[S^c])$, Circles = $(Q_p[S], Q_p[S^l])$.

An allocation problem

- Problem description:
  - Consider the loss portfolio $(X_1, \ldots, X_n)$.
  - How to allocate a given amount d among the n losses?
  - Allocation rule:
    minimize the expected aggregate shortfall:

$$\min_{\sum_{i=1}^{n} d_i = d} \mathrm{E}\left(\sum_{i=1}^{n} \left[(X_i - d_i)_+ \right] \right).$$

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An allocation problem

- Solution of the minimization problem:
  - Let $S = X_1 + \cdots + X_n$ and $S^c = X_1^c + \cdots + X_n^c$.
  - For all d_i with $\sum_{i=1}^n d_i = d$, we have

$$\mathrm{E}\left[(S^c - d)_+\right] \leq \sum_{i=1}^{n} \mathrm{E}\left[(X_i - d_i)_+\right].$$

  - As

$$\mathrm{E}\left[(S^c - d)_+\right] = \sum_{i=1}^{n} \mathrm{E}\left[\left(X_i - F_{X_i}^{-1}\left[F_{S^c}(d)\right]\right)_+\right],$$

the optimal allocation rule is given by

$$d_i^* = F_{X_i}^{-1}\left[F_{S^c}(d)\right].$$

Asian options

(Dhaene, Denuit, Goovaerts, Kaas & Vyncke, 2002)

- A European style arithmetic Asian call option:
  $\{A_t\} =$ price process of underlying asset, T = exercise date,
  $n =$ number of averaging dates, $K =$ exercise price.

$$\text{Pay-off at } T = \left(\frac{1}{n} \sum_{i=0}^{n-1} A_{T-i} - K \right)_+$$

- Arbitrage-free time-0 price:

$$\mathrm{AC}\,(n,\ K,\ T) = e^{-\delta T}\ \mathrm{E}\left[\left(\frac{1}{n} \sum_{i=0}^{n-1} A_{T-i} - K \right)_+ \right],$$

where $\delta =$ risk free interest rate and E is evaluated wrt Q.

Asian options

- The comonotonic upper bound:

$$\mathrm{AC}\,(n,\ K,\ T) \leq e^{-\delta T}\ \mathrm{E}\left[\left(\frac{1}{n}\sum_{i=0}^{n-1} A^c_{T-i} - K\right)_+\right]$$

$$= \frac{e^{-\delta T}}{n}\sum_{i=0}^{n-1}\mathrm{E}\left[\left(A_{T-i} - F^{-1}_{A_{T-i}}\left(F_{S^c}\left(nK\right)\right)\right)_+\right]$$

- The upper bound in terms of European calls:

$$\mathrm{AC}(n, K, T) \leq \sum_{i=0}^{n-1}\frac{e^{-\delta i}}{n}\ \mathrm{EC}\left(K_i^*,\ T-i\right)$$

with $K_i^* = F^{-1}_{A_{T-i}}\left(F_{S^c}\left(nK\right)\right)$.

Asian options

- Static super-replicating strategies: (Albrecher et al., 2005)
  - At time 0, for $i = 1, \ldots, n$,, buy $\frac{e^{-\delta i}}{n}$ European calls $\mathrm{EC}(K_i,\ T - i)$ with $\frac{1}{n}\sum_{i=0}^{n-1} K_i = K$.
  - Hold these European calls until expiration.
  - Invest their payoffs at expiration at the risk-free rate.
  - Payoff at T:

$$\frac{1}{n}\sum_{i=0}^{n-1}(A_{T-i} - K_i)_+ \geq \left(\frac{1}{n}\sum_{i=0}^{n-1} A_{T-i} - K\right)_+$$

  - Price at time 0:

$$\frac{1}{n}\sum_{i=0}^{n-1} e^{-\delta i}\,\mathrm{EC}(K_i, T - i) \geq \mathrm{AC}(n, K, T)$$

Asian options

- The cheapest super-replicating strategy:
  - The price $\frac{1}{n}\sum_{i=0}^{n-1} e^{-\delta i}\,\mathrm{EC}(K_i, T-i)$ of the super-replicating strategy is minimized for

$$K_i^* = F_{A_{T-i}}^{-1}(F_{S^c}(nK)).$$

  - The optimal strategy corresponds to the comonotonic upper bound.
- Remarks:
  - Similarly, comonotonic bounds can be derived for basket options (Deelstra et al., 2004).
  - The K_i^* can be determined from the European call prices observed in the market.
  - The model-free approach can be generalized to the case of a finite number of exercise prices (Hobson et al., 2005).

Asian options

- Numerical illustration in a Black & Scholes setting:
  - Risk-free interest rate = $e^{\delta} - 1 = 9\%$ per year,
  - $\{A_t\}$: geometric Brownian motion with $A_0 = 100$ and volatility per year $\sigma = 0.2$,
  - $n =$ 10 days, T = day 120.

K	LB	MC (s.e.$\times 10^4$)	UB
80	22.1712	22.1712 (0.85)	22.1735
90	13.0085	13.0083 (0.81)	13.0232
100	5.8630	5.8629 (0.75)	5.8934
110	1.9169	1.9168 (0.59)	1.9442
120	0.4534	0.4533 (0.33)	0.4665

Strategic portfolio selection

(Dhaene, Vanduffel, Goovaerts, Kaas & Vyncke, 2005)

- Provisions for future liabilities:
  - $\alpha_1, \alpha_2, ..., \alpha_n$: positive payments, due at times $1, 2, ..., n$.
  - $R =$ initial provision to be established at time 0.

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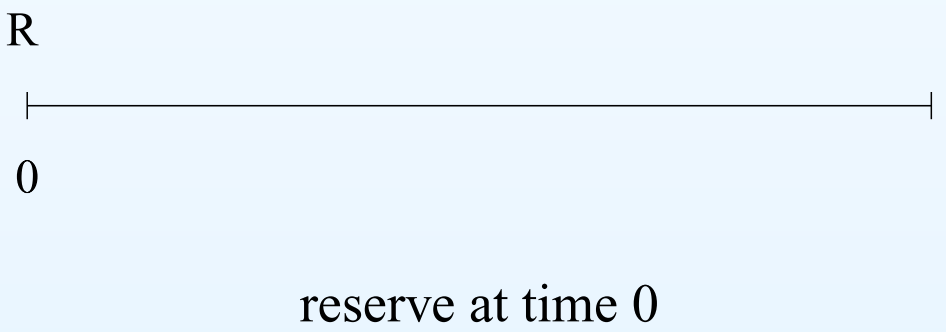

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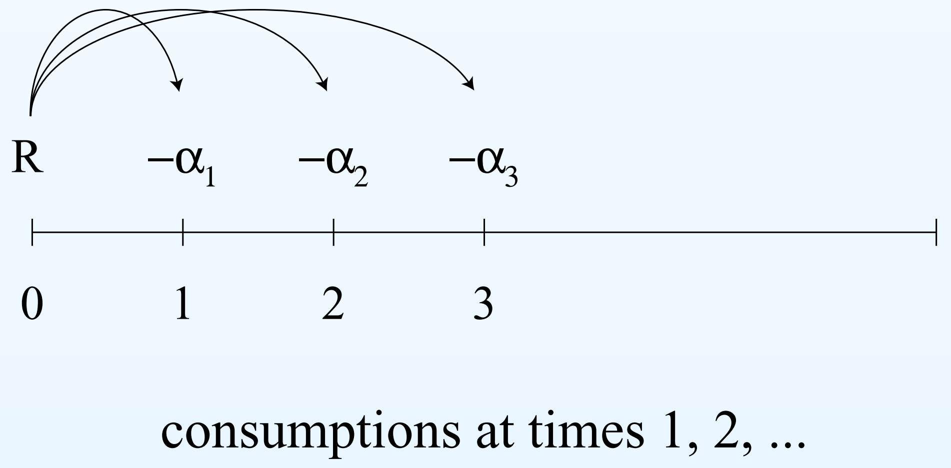


Strategic portfolio selection

- Investment strategy i, $(i = 1, \cdots, n)$:
  - Yearly returns: $(Y_1^{(i)}, \cdots, Y_n^{(i)})$.
  - The stochastic provision:

$$S^{(i)} = \sum_{j=1}^{n} \alpha_j \, e^{-\left(Y_1^{(i)} + Y_2^{(i)} + \cdots + Y_j^{(i)}\right)}.$$

  - The provision principle:

$$R_0^{(i)} = \rho_g \left[S^{(i)} \right].$$

  - Available provision at time j:

$$R_j^{(i)} = R_{j-1}^{(i)} \, e^{Y_j^{(i)}} - \alpha_j.$$

Strategic portfolio selection

- The optimal investment strategy:
  - $(i^*,\ R_0^*)$ follows from

$$R_0^* = \min_i R_0^{(i)} = \min_i \rho_g \left[S^{(i)} \right].$$

  - Avoid simulation by considering comonotonic approximations for $S^{(i)}$.
  - Example: the quantile provision principle:

$$R_0^{(i)} = Q_p \left[S^{(i)} \right] = \inf \left\{ R_0 \mid \Pr \left(R_n^{(i)} \geq 0 \right) \geq p \right\}.$$

Strategic portfolio selection: numerical example

- The Black-Scholes framework:
  - 1 riskfree asset: $\delta = 0.03$
  - 2 risky assets:

$$\left(\mu^{(1)},\ \sigma^{(1)}\right) = (0.06,\ 0.10)$$
$$\left(\mu^{(2)},\ \sigma^{(2)}\right) = (0.10,\ 0.20)$$

with

$$\text{Corr}\left[Y_k^{(1)}, Y_k^{(2)}\right] = 0.5$$

- Constant mix strategies: $\underline{\pi} = (\pi_1,\ \pi_2)$
  - $\pi_i =$ (time-independent) fraction invested in risky asset i,
  - $1 - \sum_{i=1}^{2} \pi_i =$ fraction invested in riskfree asset.

Strategic portfolio selection: numerical example

- Yearly consumptions: $\alpha_1 = \ldots = \alpha_{40} = 1$.
- Stochastic provision:

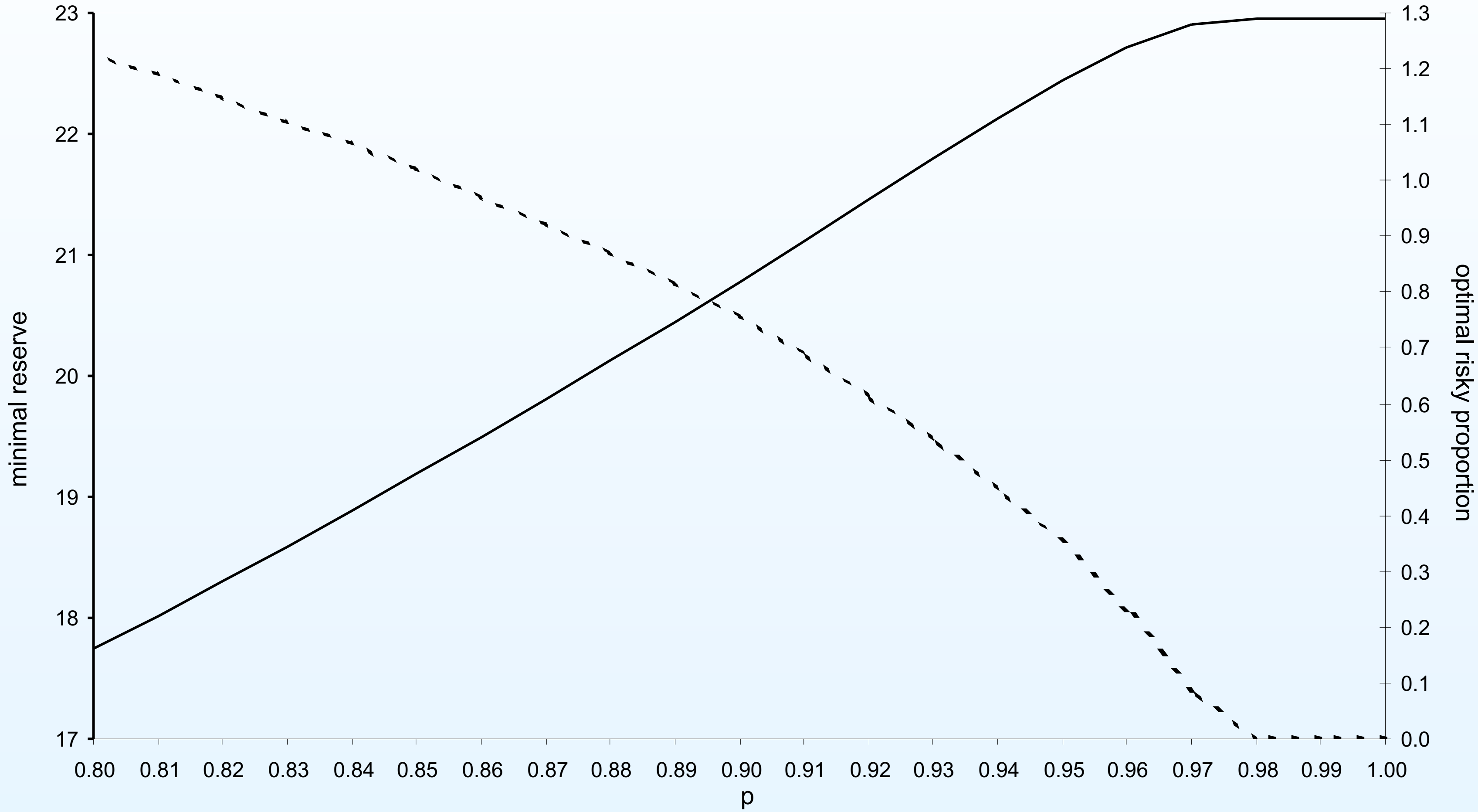
$$S\left(\underline{\pi}\right) = \sum_{i=1}^{40} e^{-(Y_1(\underline{\pi})+Y_2(\underline{\pi})+\cdots+Y_i(\underline{\pi}))}.$$

- Optimal investment strategy: $R_0^* = \min_{\underline{\pi}} Q_p \left[S\left(\underline{\pi}\right)\right]$.
- Approximation:

$$R_0 = \min_{\alpha} Q_p \left[S\left(\alpha\, \underline{\pi}^{(t)}\right)\right] \approx \min_{\alpha} Q_p \left[S^l\left(\alpha\, \underline{\pi}^{(t)}\right)\right].$$

with $\underline{\pi}^{(t)} = \left(\frac{5}{9}, \frac{4}{9}\right)$ and $\alpha =$ proportion invested in $\underline{\pi}^{(t)}$.

Strategic portfolio selection: numerical example



Solid line (left scale): minimal initial provision R_0^l as a function of p.

Dashed line (right scale): optimal proportion invested in $\underline{\pi}^{(t)}$, as a function of p.

Generalizations

- Provisions for random future liabilities:
  Goovaerts et al. (2000), Hoedemakers et al. (2003, 2005), Ahcan et al. (2004).
- The ’final wealth problem’:
  Dhaene et al. (2005).
- Stochastic sums:
  Hoedemakers et al. (2005).
- Positive and negative payments:
  Vanduffel et al. (2005).
- Other distributions:
  Albrecher et al. (2005), Valdez et al. (2005).

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