Risk Measures, Stochastic Orders and Comonotonicity

Jan Dhaene
Sums of r.v.’s

- Many problems in risk theory involve sums of r.v.’s:

\[ S = X_1 + X_2 + \cdots + X_n. \]

- Standard techniques for (approximate) evaluation of the d.f. of \( S \) are: convolution, moment-based approximations, De Pril’s recursion, Panjer’s recursion.

- Assuming independence of the \( X_i \) is convenient but not always appropriate.

- The copula approach: (Frees & Valdez, 1998)

\[ \Pr [X_1 \leq x_1, \cdots, X_n \leq x_n] = C [F_{X_1}(x_1), \ldots, F_{X_n}(x_n)]. \]
Sums of r.v.’s

- **Problem to solve:**
  Determine risk measures related to $S = X_1 + X_2 + \ldots + X_n$ in case $F_{X_i}(x)$ known, $C$ complicated / unknown, positive dependence.

- **How to solve?**
  - Derive stochastic lower and upper bounds for $S$:
    $$S^l \preceq S \preceq S^u.$$  
  - Involves risk measurement, stochastic orderings, choice under risk.
Example: A life annuity

- Consider a life annuity $a_x$ on $(x)$ with present value

$$S = X_1 + X_2 + \cdots + X_n,$$

where

$$X_j = \begin{cases} 
0 & : T(x) \leq j, \\
v^j & : T(x) > j.
\end{cases}$$

- All $X_j$ are increasing functions of $T(x)$

$\implies (X_1, X_2, \cdots, X_n)$ is **comonotonic**.
Example: A risk sharing scheme

- Consider a loss \( S \geq 0 \) that is covered by \( n \) parties:

\[
S = X_1 + X_2 + \cdots + X_n.
\]

- \( X_j \) is the layer with deductible \( a_j \) and maximal payment \((a_{j+1} - a_j)\):

\[
X_j = \begin{cases} 
0 & : 0 \leq S \leq a_j \\
S - a_j & : a_j < S \leq a_{j+1} \\
a_{j+1} - a_j & : S > a_{j+1},
\end{cases}
\]

with \( a_0 = 0 \) and \( a_{n+1} = \infty \).

- The layers \( X_j \) are comonotonic.

- Many risk sharing schemes lead to partial risks that are comonotonic.
Example: A portfolio of pure endowments

- Consider a portfolio \((X_1, X_2, \cdots, X_n)\) of \(m\)-year pure endowment insurances.

- \(X_j\): claim amount of policy \(j\) at time \(m\):

\[
X_j = \begin{cases} 
0 & : T(x) \leq m, \\
1 & : T(x) > m. 
\end{cases}
\]

- Assumption: all \(X_i\) are i.i.d. and \(X_i \overset{d}{=} X\).
Example: A portfolio of pure endowments

- Let

\[ S = (X_1 + X_2 + \cdots + X_n) \, Y, \]

where \( Y \) is the stochastic discount factor over \([0, m]\).

- Assumption: \( X_i \) and \( Y \) are mutually independent.

- Risk pooling reduces the actuarial risk, not the financial risk:

\[
\text{Var}\left[ \frac{S}{n} \right] = \frac{\text{Var}[X]}{n} \, E[Y^2] + (E[X])^2 \, \text{Var}[Y] 
\]

\[ \rightarrow (E[X])^2 \, \text{Var}[Y]. \]

- The terms \( X_i Y \) are 'conditionally comonotonic'.
Example: A provision for future liabilities

- Consider the liability cash-flow stream \((\alpha_1, 1), (\alpha_2, 2), \ldots, (\alpha_n, n)\).

- The provision is invested such that it generates a cumulative log-return \(Y(i)\) over the period \((0, i)\).

- The provision is determined as \(R[S]\) with

\[
S = \sum_{i=1}^{n} \alpha_i e^{-Y(i)}.
\]

- The cumulative returns \(Y(i)\) are 'locally quasi-comonotonic'.

- Illustration: i.i.d. normal yearly returns.
Cumulative returns
Cumulative returns
Comonotonicity

- **Definition**: \((X_1, \cdots, X_n)\) is **comonotonic** if there exists a r.v. \(Z\) and increasing functions \(f_1, \cdots, f_n\) such that

\[
(X_1, \cdots, X_n) \overset{d}{=} (f_1(Z), \cdots, f_n(Z)) .
\]

- Determining the d.f. of \((X_1, \cdots, X_n)\) is a one-dimensional problem.

- Comonotonicity is very strong positive dependency structure.

- Adding comonotonic r.v.’s produces no diversification: If all \(X_i\) are identically distributed and comonotonic, then

\[
\frac{X_1 + \cdots + X_n}{n} \overset{d}{=} X_1 .
\]
An example of comonotonic r.v.’s

- Consider \((X, Y, Z)\) with
  
  \(X \sim\) Uniform on \((0, \frac{1}{2}) \cup (1, \frac{3}{2})\)
  
  \(Y \sim\) Beta (2,2)
  
  \(Z \sim\) Normal (0,1).

- \((X, Y, Z)\) mutually independent
An example of comonotonic r.v.’s

• Consider \((X, Y, Z)\) with
  \[ X \sim \text{Uniform on } (0, \frac{1}{2}) \cup (1, \frac{3}{2}) \]
  \[ Y \sim \text{Beta (2,2)} \]
  \[ Z \sim \text{Normal (0,1)} \].

• \((X, Y, Z)\) comonotonic
Sums of comonotonic r.v.’s

- **Notation**: \((X_1^c, \ldots, X_n^c)\) is comonotonic and has same marginals as \((X_1, \ldots, X_n)\).

- \(S^c = X_1^c + X_2^c + \cdots + X_n^c\).

- **Quantiles of** \(S^c\):
  \[
  F_{S^c}^{-1}(p) = \sum_{i=1}^{n} F_{X_i}^{-1}(p).
  \]

- **Distribution function of** \(S^c\):
  \[
  \sum_{i=1}^{n} F_{X_i}^{-1}\left[F_{S^c}(x)\right] = x.
  \]
Sums of comonotonic r.v.’s

- **Stop-loss premiums of** $S^c$:  
  (Dhaene, Wang, Young, Goovaerts, 2000)

$$E [S^c - d]_+ = \sum_{i=1}^{n} E [(X_i - d_i)_+]$$

with

$$d_i = F_{X_i}^{-1} [F_{S^c}(d)].$$

- **Application**: (Jamshidian, 1989)  
  In the Vasicek (1977) model, the price of a European call option on a coupon bond = sum of the prices of European options on zero coupon bonds.
Theories of choice under risk

- **Wealth level vs. probability level:**
  \[
  E[X] = \int_0^1 F_X^{-1}(1 - q) \, dq.
  \]

- **Utility theory:** (von Neumann & Morgenstern, 1947)
  \[
  E[u(X)] = \int_0^1 u \left[ F_X^{-1}(1 - q) \right] \, dq
  \]
  with \( u(x) \) a utility function.

- **Dual theory of choice under risk:** (Yaari, 1987)
  \[
  \rho_f[X] = \int_0^1 F_X^{-1}(1 - q) \, df(q)
  \]
  with \( f(q) \) a distortion function.
Theories of choice under risk

• Choice under risk:
  ◦ Prefer wealth $Y$ over wealth $X$ if
    \[ E[u(X)] \leq E[u(Y)]. \]
  ◦ Prefer wealth $Y$ over wealth $X$ if
    \[ \rho_f [X] \leq \rho_f [Y]. \]

• Additivity:
  ◦ If $u(0) = 0$ and $\Pr [X \neq 0, Y \neq 0] = 0$, then
    \[ E[u(X + Y)] = E[u(X)] + E[u(Y)]. \]
  ◦ If $X$ and $Y$ are comonotonic, then
    \[ \rho_f [X + Y] = \rho_f [X] + \rho_f [Y]. \]
Risk measures

- **Definition:**
  = mapping from the set of quantifiable losses to the real line:

  \[ X \rightarrow R[X] \, . \]

- Have been investigated extensively in the literature:
  - Huber (1981):
    upper expectations,
  - Goovaerts, De Vylder & Haezendonck (1984):
    premium principles,
  - Artzner, Delbaen, Eber & Heath (1999):
    coherent risk measures.
Construction of risk measures

- The equivalent expected utility principle:
  \[ u(w) = \mathbb{E}[u(w + R[X] - X)]. \]

- The equivalent distorted expectation principle:
  \[ \rho_f[w] = \rho_f[w + R[X] - X]. \]

This leads to distortion risk measures (Wang, 1996):

\[ R[X] = \rho_g[X] \]

with \( g(q) = 1 - f(1 - q) \).
Distortion risk measures

\[ E[X] = I - II \]
Distortion risk measures

\[ E[X] = I - (\Pi + \Pi') \]

\[ \rho_g [X] = (I + I') - \Pi \geq E[X] \]
Examples of distortion risk measures

- **Value-at-Risk**: $X \rightarrow \text{VaR}_p[X] = F_X^{-1}(p) = Q_p[X]$. 

\[ g(x) = I(x > 1 - p), \quad 0 \leq x \leq 1. \]
Examples of distortion risk measures

- **Tail Value-at-Risk:** \( X \rightarrow \text{TVaR}_p(X) = \frac{1}{1-p} \int_p^1 \text{VaR}_q[X] \, dq. \)

\[
g(x) = \min \left( \frac{x}{1-p}, 1 \right), \quad 0 \leq x \leq 1.
\]
Concave distortion risk measures

- $\rho_g [.]$ is a concave distortion risk measure if $g$ is concave.
- TVaR$_p$ is concave, VaR$_p$ not.
- Concave distortion risk measures are subadditive:
  \[ \rho_g [X + Y] \leq \rho_g [X] + \rho_g [Y]. \]

- Optimality of TVaR$_p$:
  \[ \text{TVaR}_p[X] = \min \{ \rho_g([X] \mid g \text{ is concave and } \rho_g \geq \text{VaR}_p) \}. \]

- Optimality of VaR$_p$: (Artzner et al. 1999)
  \[ \text{VaR}_p[X] = \inf \{ \rho([X] \mid \rho \text{ is coherent and } \rho \geq \text{VaR}_p) \}. \]
Optimality of VaR

- Consider a loss $X$ and a solvency capital requirement $R[X]$.
- Measuring the insolvency risk:

$$ (X - R[X])_+ \rightarrow \mathbb{E} [(X - R[X])_+] . $$

- How to choose $R[X]$?
  - $\mathbb{E} [(X - R[X])_+]$ should be small
  $\Rightarrow$ choose $R[X]$ large enough.
  - Capital has a cost
  $\Rightarrow$ $R[X]$ should be small enough.
Optimality of VaR\(_p\)

- The optimal capital requirement: 
  \( R[X] \) is determined as the minimizer (with respect to \( d \)) of 
  \[ E[(X - d)_+] + d \varepsilon, \quad 0 < \varepsilon < 1. \]

- Solution: 
  \[ R[X] = \text{VaR}_{1-\varepsilon}[X]. \]

- The minimum is given by \( \varepsilon \text{ TVaR}_{1-\varepsilon}[X] \).

- Geometric proof (for \( \text{VaR}_{1-\varepsilon}[X] > 0 \)): 

\[ \]
Optimality of VaR$_p$

\[ E[(X - d)_+] + d \varepsilon \text{ with } d = Q_{1-\varepsilon}[X] \]
Optimality of VaR$_p$

\[ \mathbb{E}[(X - d)_+] + d \in \text{ with } d < Q_{1-\varepsilon}[X] \]
Optimality of $\text{VaR}_p$

$$E[(X - d)_+] + d \varepsilon \text{ with } d > Q_{1-\varepsilon}[X]$$
Can a risk measure be too subadditive?

(Dhaene, Laeven, Vanduffel, Darkiewicz, Goovaerts, 2005)

- For losses $X$ and $Y$, we have that

$$
E[(X + Y - R[X] - R[Y])_+] 
\leq E[(X - R[X])_+] + E[(Y - R[Y])_+].
$$

- **Splitting** increases the insolvency risk
  $\Rightarrow$ the risk measure used to determine the required solvency capital should be subadditive enough.

- **Merging** decreases the insolvency risk
  $\Rightarrow$ subadditivity of the capital requirement is allowed *to some extent.*
  $\Rightarrow$ the capital requirement can be *too subadditive* if no constraint is imposed on the subadditivity.
Can a risk measure be too subadditive?

- **The regulator’s condition:**

\[
E \left( (X + Y - R[X + Y])_+ \right) + \varepsilon R[X + Y] \\
\leq E \left( (X - R[X])_+ \right) + E \left( (X - R[X])_+ \right) + \varepsilon \left( R[X] + R[Y] \right)
\]

- VaR$_{1-\varepsilon} [\cdot]$ fulfills the regulator’s condition.
- Any subadditive $R[\cdot] \geq$ VaR$_{1-\varepsilon} [\cdot]$ fulfills the regulator’s condition.
- Mark(ovitz), 1959:
  ’We might decide that in one context one basic set of principles is appropriate, while in another context a different set of principles should be used.’
Can a risk measure be too subadditive?

- **The regulator’s condition:**

\[
E \left[ (X + Y - R[X + Y])_+ \right] + \varepsilon R[X + Y] \\
\leq E \left[ (X - R[X])_+ \right] + E \left[ (X - R[X])_+ \right] + \varepsilon (R[X] + R[Y])
\]

- VaR$_{1-\varepsilon}[^{\cdot}]$ fulfills the regulator’s condition.
- Any subadditive $R[^{\cdot}] \geq$ VaR$_{1-\varepsilon}[^{\cdot}]$ fulfills the regulator’s condition.
- Markowitz, 1959: ‘We might decide that in one context one basic set of principles is appropriate, while in another context a different set of principles should be used.’
Stochastic orderings - Upper and lower tails

- $E \left[ (X - d)_+ \right] = \text{surface above the d.f., from } d \text{ on.}$
- $E \left[ (d - X)_+ \right] = \text{surface below the d.f., from } -\infty \text{ to } d.$
Stochastic orderings - Upper and lower tails

- $\mathbb{E}[(X - d)_+]$ = surface above the d.f., from $d$ on.
- $\mathbb{E}[(d - X)_+]$ = surface below the d.f., from $-\infty$ to $d$. 

![Diagram showing the relationship between $F_X(x)$ and $\mathbb{E}[(X-d)_+]$]
Stochastic orderings - Upper and lower tails

- $E[(X - d)_+] = \text{surface above the d.f., from } d \text{ on.}$
- $E[(d - X)_+] = \text{surface below the d.f., from } -\infty \text{ to } d.$
Convex order

- Definition:
  \[ X \leq_{cx} Y \Leftrightarrow \text{any tail of } Y \text{ exceeds the respective tail of } X. \]

- Represents common preferences of risk averse decision makers between r.v.’s with equal means.

- Characterization in terms of distortion risk measures:
  (Wang & Young, 1998)
  \[ X \leq_{cx} Y \Leftrightarrow \mathbb{E}[X] = \mathbb{E}[Y] \text{ and } \rho_g [X] \leq \rho_g [Y] \text{ for all concave } g. \]
Stochastic order bounds for sums of dependent r.v.’s

- **Theorem:** (Kaas et al., 2000)

  For any \((X_1, \cdots, X_n)\) and any \(\Lambda\), we have that

  \[
  \sum_{i=1}^{n} \mathbb{E}[X_i | \Lambda] \leq_{cx} \sum_{i=1}^{n} X_i \leq_{cx} \sum_{i=1}^{n} X_i^c
  \]

- **Notation:** \(S^l \leq_{cx} S \leq_{cx} S^c\).

- Assume that all \(\mathbb{E}[X_i | \Lambda]\) are \(\nearrow\) functions of \(\Lambda\)
  \(\Rightarrow S^l\) is a comonotonic sum.

- **Why use these comonotonic bounds?**
  - One-dimensional stochasticity.
  - \(\rho_g [S^l]\) and \(\rho_g [S^c]\) are easy to calculate.
  - If \(g\) is concave, then \(\rho_g [S^l] \leq \rho_g [S] \leq \rho_g [S^c]\).
On the choice of $\Lambda$

(Vanduffel et al., 2004)

- Let

$$S = \sum_{i=1}^{n} \alpha_i e^{-Y(i)} \quad \text{and} \quad S^l = \sum_{i=1}^{n} \alpha_i \mathbb{E}\left[ e^{-Y(i)} \mid \Lambda \right]$$

with $\alpha_i > 0$ and $(Y_1, \cdots, Y_n)$ normal.

- First order approximation for $\text{Var}[S^l]$: 

$$\text{Var}[S^l] \approx \text{Corr}^2 \left[ \sum_{i=1}^{n} \alpha_i \mathbb{E}[e^{-Y(i)}]Y(i), \Lambda \right] \text{Var} \left[ \sum_{i=1}^{n} \alpha_i \mathbb{E}[e^{-Y(i)}]Y(i) \right].$$

- Optimal choice for $\Lambda$:

$$\Lambda = \sum_{i=1}^{n} \alpha_i \mathbb{E}\left[ e^{-Y(i)} \right] Y(i).$$
The continuous perpetuity

- **Local comonotonicity**: Let $B(\tau)$ be a standard Wiener process. The accumulated returns
  
  $$\exp[\mu\tau + \sigma B(\tau)] \text{ and } \exp[\mu(\tau + \Delta\tau) + \sigma B(\tau + \Delta\tau)]$$

  are 'almost comonotonic'.

- **The continuous perpetuity**: (Dufresne, 1989; Milevsky, 1997)

  $$S = \int_0^\infty \exp[-\mu\tau - \sigma B(\tau)] \, d\tau$$

  has a reciprocal Gamma distribution.
The continuous perpetuity

- **Numerical illustration:** $\mu = 0.07$ and $\sigma = 0.1$.

\[
\text{Squares } = (Q_p[S], Q_p[S^c]), \quad \text{Circles } = (Q_p[S], Q_p[S^l]).
\]
An allocation problem

- **Problem description**:
  - Consider the loss portfolio \((X_1, \ldots, X_n)\).
  - How to allocate a given amount \(d\) among the \(n\) losses?
  - **Allocation rule**: minimize the expected aggregate shortfall:

\[
\min_{\sum_{i=1}^{n} d_i = d} \mathbb{E}\left( \sum_{i=1}^{n} \left( X_i - d_i \right)_+ \right).
\]
An allocation problem

- Solution of the minimization problem:
  - Let $S = X_1 + \cdots + X_n$ and $S^c = X_1^c + \cdots + X_n^c$.
  - For all $d_i$ with $\sum_{i=1}^{n} d_i = d$, we have
    \[
    \mathbb{E} \left[ (S^c - d)_+ \right] \leq \sum_{i=1}^{n} \mathbb{E} \left[ (X_i - d_i)_+ \right].
    \]
  - As
    \[
    \mathbb{E} \left[ (S^c - d)_+ \right] = \sum_{i=1}^{n} \mathbb{E} \left[ (X_i - F_{X_i}^{-1} [F_{S^c}(d)])_+ \right],
    \]

the optimal allocation rule is given by
\[
    d^*_i = F_{X_i}^{-1} [F_{S^c}(d)].
\]
Asian options

(Dhaene, Denuit, Goovaerts, Kaas & Vyncke, 2002)

- A European style arithmetic Asian call option:
  \[ \{ A_t \} = \text{price process of underlying asset}, \ T = \text{exercise date}, \ n = \text{number of averaging dates}, \ K = \text{exercise price}. \]

  Pay-off at \[ T = \left( \frac{1}{n} \sum_{i=0}^{n-1} A_{T-i} - K \right) + \]

- Arbitrage-free time-0 price:

  \[ AC(n, K, T) = e^{-\delta T} \ E \left[ \left( \frac{1}{n} \sum_{i=0}^{n-1} A_{T-i} - K \right) + \right], \]

  where \( \delta = \text{risk free interest rate} \) and \( E \) is evaluated wrt \( Q \).
Asian options

- The comonotonic upper bound:

\[
AC(n, K, T) \leq e^{-\delta T} \left( \frac{1}{n} \sum_{i=0}^{n-1} A_{T-i}^c - K \right) + \sum_{i=0}^{n-1} e^{-\delta_i} \left( A_{T-i} - F_{A_{T-i}}^{-1}(FS_c(nK)) \right) +
\]

- The upper bound in terms of European calls:

\[
AC(n, K, T) \leq \sum_{i=0}^{n-1} \frac{e^{-\delta_i}}{n} EC(K^*_i, T - i)
\]

with \(K^*_i = F_{A_{T-i}}^{-1}(FS_c(nK))\).
Asian options

• **Static super-replicating strategies**: (Albrecher et al., 2005)
  
  ◦ At time 0, for \( i = 1, \ldots, n \), buy \( \frac{e^{-\delta_i}}{n} \) European calls \( EC(K_i, T - i) \) with \( \frac{1}{n} \sum_{i=0}^{n-1} K_i = K \).
  
  ◦ Hold these European calls until expiration.
  
  ◦ Invest their payoffs at expiration at the risk-free rate.
  
  ◦ **Payoff at** \( T \):
    \[
    \frac{1}{n} \sum_{i=0}^{n-1} (A_{T-i} - K_i)_+ \geq \left( \frac{1}{n} \sum_{i=0}^{n-1} A_{T-i} - K \right)_+
    \]
  
  ◦ **Price at time 0**:
    \[
    \frac{1}{n} \sum_{i=0}^{n-1} e^{-\delta_i} EC(K_i, T - i) \geq AC(n, K, T)
    \]
Asian options

- The cheapest super-replicating strategy:
  - The price \( \frac{1}{n} \sum_{i=0}^{n-1} e^{-\delta_i} EC(K_i, T - i) \) of the super-replicating strategy is minimized for
    \[
    K_i^* = F_{A_{T-i}}^{-1}(F_{Sc}(nK)).
    \]
  - The optimal strategy corresponds to the comonotonic upper bound.

- Remarks:
  - Similarly, comonotonic bounds can be derived for basket options (Deelstra et al., 2004).
  - The \( K_i^* \) can be determined from the European call prices observed in the market.
  - The model-free approach can be generalized to the case of a finite number of exercise prices (Hobson et al., 2005).
Asian options

- Numerical illustration in a Black & Scholes setting:
  - Risk-free interest rate $= e^{\delta} - 1 = 9\%$ per year,
  - $\{A_t\}$: geometric Brownian motion with $A_0 = 100$ and volatility per year $\sigma = 0.2$,
  - $n = 10$ days, $T = \text{day 120}$.

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Strategic portfolio selection

(Dhaene, Vanduffel, Goovaerts, Kaas & Vyncke, 2005)

- **Provisions for future liabilities:**
  - $\alpha_1, \alpha_2, \ldots, \alpha_n$: positive payments, due at times $1, 2, \ldots, n$.
  - $R =$ initial provision to be established at time 0.
Strategic portfolio selection

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\[
\text{R}
\]

\[
\text{0}
\]

reserve at time 0
Strategic portfolio selection

(Dhaene, Vanduffel, Goovaerts, Kaas & Vyncke, 2005)

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  - $R =$ initial provision to be established at time 0.

consumptions at times $1, 2, \ldots$
Strategic portfolio selection

- **Investment strategy** \( i, (i = 1, \cdots, n) \):
  - Yearly returns: \( (Y_1^{(i)}, \cdots, Y_n^{(i)}) \).
  - The stochastic provision:
    \[
    S^{(i)} = \sum_{j=1}^{n} \alpha_j \ e^{-\left( Y_1^{(i)} + Y_2^{(i)} + \cdots + Y_j^{(i)} \right)}.
    \]
  - The provision principle:
    \[
    R_0^{(i)} = \rho_g \left[ S^{(i)} \right].
    \]
  - Available provision at time \( j \):
    \[
    R_j^{(i)} = R_{j-1}^{(i)} \ e^{Y_j^{(i)}} - \alpha_j.
    \]
Strategic portfolio selection

- The optimal investment strategy:
  - \((i^*, R_{0}^*)\) follows from
    \[
    R_{0}^* = \min_i R_{0}^{(i)} = \min_i \rho_g \left[ S^{(i)} \right].
    \]
  - Avoid simulation by considering comonotonic approximations for \(S^{(i)}\).
  - Example: the quantile provision principle:
    \[
    R_{0}^{(i)} = Q_p \left[ S^{(i)} \right] = \inf \left\{ R_0 \mid \Pr \left( R_{n}^{(i)} \geq 0 \right) \geq p \right\}.
    \]
Strategic portfolio selection: numerical example

- **The Black-Scholes framework:**
  - 1 riskfree asset: $\delta = 0.03$
  - 2 risky assets:
    \[
    \begin{align*}
    (\mu^{(1)}, \sigma^{(1)}) &= (0.06, 0.10) \\
    (\mu^{(2)}, \sigma^{(2)}) &= (0.10, 0.20)
    \end{align*}
    \]
    with
    \[
    \text{Corr} \left[ Y^{(1)}_k, Y^{(2)}_k \right] = 0.5
    \]

- **Constant mix strategies:** $\pi = (\pi_1, \pi_2)$
  - $\pi_i$ = (time-independent) fraction invested in risky asset $i$,
  - $1 - \sum_{i=1}^{2} \pi_i$ = fraction invested in riskfree asset.
Strategic portfolio selection: numerical example

- **Yearly consumptions**: \( \alpha_1 = \ldots = \alpha_{40} = 1 \).

- **Stochastic provision**:

\[
S(\pi) = \sum_{i=1}^{40} e^{-(Y_1(\pi)+Y_2(\pi)+\ldots+Y_i(\pi))}.
\]

- **Optimal investment strategy**: \( R^*_0 = \min_{\pi} Q_p [S(\pi)] \).

- **Approximation**:

\[
R_0 = \min_{\alpha} Q_p \left[ S \left( \alpha \pi^{(t)}(t) \right) \right] \approx \min_{\alpha} Q_p \left[ S^{l} \left( \alpha \pi^{(t)}(t) \right) \right].
\]

with \( \pi^{(t)} = \left( \frac{5}{9}, \frac{4}{9} \right) \) and \( \alpha = \text{proportion invested in } \pi^{(t)} \).
Solid line (left scale): minimal initial provision $R^l_0$ as a function of $p$.
Dashed line (right scale): optimal proportion invested in $\pi^{(t)}$, as a function of $p$. 

Strategic portfolio selection: numerical example
Generalizations

• **Provisions for random future liabilities:**

• **The ’final wealth problem’:**
  Dhaene et al. (2005).

• **Stochastic sums:**
  Hoedemakers et al. (2005).

• **Positive and negative payments:**
  Vanduffel et al. (2005).

• **Other distributions:**
  Albrecher et al. (2005), Valdez et al. (2005).
References (www.kuleuven.be/insurance)

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